Surprise: The Big Bang isn't the beginning of the universe anymore

Překvapení: Velký třesk už není začátkem vesmíru. (Úžasnýý. V Česku překvapený není ve fyzikální komunitě nikdo !)

https://news.sciandnature.com/2023/03/surprise-big-bang-isnt-beginning-of.html?m=1&fbclid=IwAR0MvkPyA9LYIYztNnTBz5FES8U_lkF7QWLB5z1TypfThXqGjELbworuaO8

Bruno Bento, a physicist who studies the nature of time at the University of Liverpool

(01)- "Reality has so many things that most people would associate with sci-fi or even fantasy," said Bruno Bento, a physicist who studies the nature of time at the University of Liverpool in the U.K.

In his work, he employed a new theory of quantum gravity, called causal set theory, in which space and time are broken down into discrete chunks of space-time.

At some level, there's a fundamental unit of space-time, according to this theory. Bento and his collaborators used this causal-set approach to explore the beginning of the universe. They found that it's possible that the universe had no beginning — that it has always existed into the infinite past and only recently evolved into what we call the Big Bang.

A quantum of gravity Quantum gravity is perhaps the most frustrating problem facing modern physics. We have two extraordinarily effective theories of the universe: quantum physics and general relativity.

Quantum physics has produced a successful description of three of the four fundamental forces of nature (electromagnetism, the weak force and the strong force) down to microscopic scales. General relativity, on the other hand, is the most powerful and complete description of gravity ever devised.

But for all its strengths, general relativity is incomplete. In at least two specific places in the universe, the math of general relativity simply breaks down, failing to produce reliable results: in the centers of black holes and at the beginning of the universe.

These regions are called "singularities," which are spots in space-time where our current laws of physics crumble, and they are mathematical warning signs that the theory of general relativity is tripping over itself. Within both of these singularities, gravity becomes incredibly strong at very tiny length scales.

As such, to solve the mysteries of the singularities, physicists need a microscopic description of strong gravity, also called a quantum theory of gravity. There are lots of contenders out there, including string theory and loop quantum gravity. And there's another approach that completely rewrites our understanding of space and time.

Causal set theory In all current theories of physics, space and time are continuous. They form a smooth fabric that underlies all of reality. In such a continuous space-time, two points can be as close to each other in space as possible, and two events can occur as close in time to each other as possible. “Reality has so many things that most people would associate with sci-fi or even fantasy.” Bruno Bento But another approach, called causal set theory, reimagines space-time as a series of discrete chunks, or space-time “atoms.”

This theory would place strict limits on how close events can be in space and time, since they can’t be any closer than the size of the “atom.” For instance, if you’re looking at your screen reading this, everything seems smooth and continuous.

But if you were to look at the same screen through a magnifying glass, you might see the pixels that divide up the space, and you’d find that it’s impossible to bring two images on your screen closer than a single pixel. This theory of physics excited Bento.

“I was thrilled to find this theory, which not only tries to go as fundamental as possible — being an approach to quantum gravity and actually rethinking the notion of space-time itself — but which also gives a central role to time and what it physically means for time to pass, how physical your past really is and whether the future exists already or not,” Bento told Live Science.

Beginning of time Causal set theory has important implications for the nature of time. “A huge part of the causal set philosophy is that the passage of time is something physical, that it should not be attributed to some emergent sort of illusion or to something that happens inside our brains that makes us think time passes; this passing is, in itself, a manifestation of the physical theory,” Bento said. “So, in causal set theory, a causal set will grow one ‘atom’ at a time and get bigger and bigger.”

The causal set approach neatly removes the problem of the Big Bang singularity because, in the theory, singularities can’t exist. It’s impossible for matter to compress down to infinitely tiny points — they can get no smaller than the size of a space-time atom. So without a Big Bang singularity, what does the beginning of our universe look like?

That’s where Bento and his collaborator, Stav Zalel, a graduate student at Imperial College London, picked up the thread, exploring what causal set theory has to say about the initial moments of the universe. Their work appears in a paper published Sept. 24 to the preprint database arXiv. (The paper has yet to be published in a peer-reviewed scientific journal.)

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(01)- „Realita má tolik věcí, které by si většina lidí spojovala se sci-fi nebo dokonce fantasy,“ řekl **Bruno Bento,** fyzik, který studuje povahu času na University of Liverpool *Bruno.Bento@liverpool*.ac.uk ve Spojeném království. Ve své práci **použil novou teorii kvantové gravitace, nazvanou kauzální teorie množin,** ve které jsou prostor a čas rozloženy na jednotlivé části časoprostoru. Na určité úrovni existuje podle této teorie základní jednotka časoprostoru. Čili jedna kulička, jednotková kulička z 3+1D. Zatím nic ohromujícího. Bento a jeho spolupracovníci použili tento kauzální přístup k prozkoumání počátku vesmíru. **Zjistili**, že je možné, že vesmír neměl počátek úúžasný– že vždy existoval do nekonečné minulosti úúžasný a teprve nedávno se vyvinul do toho, čemu říkáme Velký třesk. Objevná novinka.

Také popisuji 22 let Vesmír, který nevznikl ve velkém třesku →

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Kvantová gravitace

Kvantová gravitace je možná tím nejvíce frustrujícím problémem, kterému moderní fyzika čelí. Máme dvě mimořádně účinné teorie vesmíru: kvantovou fyziku a obecnou teorii relativity. Kvantová fyzika vytvořila úspěšný popis tří ze čtyř základních přírodních sil (elektromagnetismus, slabá síla a silná síla) až do mikroskopických měřítek. Obecná teorie relativity je naproti tomu nejmocnějším a nejúplnějším popisem gravitace, jaký byl kdy vymyšlen. Ale přes všechny své silné stránky je obecná teorie relativity neúplná. Nejméně na dvou konkrétních místech ve vesmíru se matematika obecné teorie relativity jednoduše porouchá a neposkytne spolehlivé výsledky: v centrech černých děr a na počátku vesmíru. Tyto oblasti se nazývají „singularity“, což jsou místa v časoprostoru, kde se naše současné fyzikální zákony hroutí, a jsou to matematické varovné signály, že teorie obecné relativity zakopává sama o sebe. V obou těchto singularitách se gravitace stává neuvěřitelně silnou na velmi malých délkách. Fyzikové jako takové potřebují k vyřešení záhad singularit mikroskopický popis silné gravitace, nazývaný také kvantová teorie gravitace. Existuje spousta uchazečů, včetně teorie strun a smyčkové kvantové gravitace. A je tu další přístup, který zcela přepisuje naše chápání prostoru a času. Teorie kauzálních množin Ve všech **současných teoriích** fyziky jsou prostor a čas spojité. Ve všech ne, v QM spojité nejsou **Tvoří hladkou tkaninu, která je základem veškeré reality.** Takže hmota nepatří do veškeré reality podle **Bruno Bento !?**, jen tkanina, síť, předivo, pavučina, rastr, časoprostor 3+3D, ano, pane ? V takto spojitém časoprostoru mohou být dva body co nejblíže u sebe v prostoru a ke dvěma událostem může dojít co nejblíže k sobě. "Realita má tolik věcí, které by si většina lidí spojovala se sci-fi nebo dokonce fantasy." **Bruno Bento** ale má jiný přístup, nazývaný teorie kauzálních množin, přetváří časoprostor jako sérii diskrétních kousků neboli časoprostorových „atomů“. Diskrétní „kousky“ nemůžou být v realitě Jsoucna-Vesmíru nic jiného než >můj< **„balíček-klubíčko sbalených dimenzí dvou veličin, Čas a Délka“.** HDV to 40 let popisuje, a 22 let na internetu. Tato teorie by stanovila přísná omezení toho, jak blízko mohou být události v prostoru a čase, protože nemohou být blíže než velikost „atomu". Pokud se například díváte na obrazovku a čtete toto, vše se zdá plynulé a plynulé. Pokud byste se však na stejnou obrazovku podívali přes lupu, mohli byste vidět pixely, http://www.hypothesis-of-universe.com/docs/c/c_040.jpg které rozdělují prostor, a zjistili byste, že je nemožné přiblížit dva obrázky na obrazovce než jeden pixel. Tato teorie fyziky Bento **nadchla**. Které fyziky například ? ? ? ; mě před 42 lety nadchla moje dvouveličinová hypotéza a… a dodnes se trápím s tím, abych odborníky přiměl jí aspoň číst. Nečetl nikdo „Byl jsem nadšený, i já… že jsem našel tuto teorii, i já která se nejen snaží jít co nejzákladnější – jde o přístup ke kvantové gravitaci a ve skutečnosti přehodnocuje pojem samotného časoprostoru – ale která také přisuzuje ústřední roli času i já (dokonce jsem použil čas jako stavební kámen k výrobě hmoty) http://www.hypothesis-of-universe.com/index.php?nav=e a tomu, co fyzikálně znamená, že čas uplyne, Čas neplyne, ale my plyneme „po čase", my se pohybujeme na „předivu časoprostoru", na síti 3+3D po dimenzi časové a tím ukrajujeme časové intervaly – **to je tok řasu**, ony intervaly, které >objekt< vykoná svým posunem „po čase, „po dimenzi časové". - - Jak prosté Sherloku = Bento, že (!) jak fyzická skutečně je vaše minulost a zda budoucnost již existuje nebo ne," **řekl Bento Live Science.**

Počátek času. Kauzální teorie množin má důležité důsledky pro povahu času. „Velká část filozofie kauzálních množin spočívá v tom, že plynutí času je něco fyzického, co by nemělo být připisováno nějakému vznikajícímu druhu iluze nebo něčemu, O.K. co se děje v našem mozku, co nás nutí si myslet, že čas plyne; toto míjení je samo o sobě projevem fyzikální teorie," řekl Bento. "Takže v teorii kauzálních množin bude kauzální množina růst jeden atom po druhém a bude se zvětšovat a zvětšovat." Přístup kauzálních množin úhledně odstraňuje problém singularity velkého třesku, protože teoreticky singularity nemohou existovat. O.K. Je nemožné, aby se hmota stlačila do nekonečně malých bodů Jistě ! – nemohou být menší než velikost atomu časoprostoru. Nový vynález, a nové pojmenování mého vlnobalíčku" z 3+3D → „ atom časoprostoru". Jak tedy vypadá počátek našeho vesmíru bez singularity velkého třesku? **Takhle** http://www.hypothesis-of-universe.com/en/index.php?nav=home. Můj popis je na 500 stranách textu k dispozici. Zde se Bento a jeho spolupracovník **Stav Zalel**, postgraduální student na Imperial College London, chopili vlákna a prozkoumali, co může teorie kauzálních množin říci o počátečních okamžicích vesmíru. Jejich práce se objevují v článku publikovaném 24. září v databázi předtisků arXiv. Tak to já neumím, zveřejňovat neumím, na to nemám kamarády v české fyzikální komunitě, aby mi s tím pomohli… spíš tak do blázince, to jo, takové chutě oni mají. Českej člověk moderní doby :…**plný nenávisti a nabubřelosti** (Příspěvek musí být ještě publikován v recenzovaném vědeckém časopise.)

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(02)- The paper examined "whether a beginning must exist in the causal set approach," Bento said. "In the original causal set formulation and dynamics, classically speaking, a causal set grows from nothing into the universe we see today.

In our work instead, there would be no Big Bang as a beginning, as the causal set would be infinite to the past, and so there's always something before." Their work implies that the universe may have had no beginning — that it has simply always existed.

What we perceive as the Big Bang may have been just a particular moment in the evolution of this always-existing causal set, not a true beginning. There's still a lot of work to be done, however. It's not clear yet if this no-beginning causal approach can allow for physical theories that we can work with to describe the complex evolution of the universe during the Big Bang.

"One can still ask whether this [causal set approach] can be interpreted in a 'reasonable' way, or what such dynamics physically means in a broader sense, but we showed that a framework is indeed possible," Bento said. "So at least mathematically, this can be done."

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(02) – Článek zkoumal, „zda v přístupu kauzálních množin musí existovat začátek," řekl Bento. Na to už mám odpověď. Mnoho let jí mám a předkládám veřejnosti. „**V původní** formulaci a dynamice kauzálních množin, klasicky řečeno, kauzální množina roste **z ničeho** do vesmíru, který dnes vidíme. V naší práci by místo toho nebyl **žádný velký třesk jako začátek**, O.K. Velký třesk pouze jako změna stavu předešlého na následný …atd., dle výkladu http://www.hypothesis-of-universe.com/index.php?nav=aa protože kauzální soubor by byl nekonečný do minulosti, a tak je tu vždy něco předtím." Jejich práce naznačuje, že vesmír možná neměl počátek – že prostě vždy existoval. To, co vnímáme jako velký třesk, mohlo být jen určitým okamžikem ve vývoji tohoto vždy existujícího kauzálního souboru, nikoli skutečným začátkem. O.K. **Bruno Bento !?**, se přiblížil k mé hypotéze a je už na tom lépe než celá dosavadní kosmologie…Stále je však potřeba udělat hodně práce. Zatím není jasné, zda tento kauzální přístup bez začátku může umožnit fyzikální teorie, se kterými můžeme pracovat, abychom popsali složitý vývoj vesmíru během Velkého třesku. Žádný složitý vývoj „ve Třesku" nebyl a ani být nemusel, viz můj výklad HDV „Stále se lze ptát, zda lze tento [příčinný souborový přístup] interpretovat ‚rozumným' způsobem nebo co taková dynamika fyzicky znamená v širším smyslu, ale ukázali jsme, že rámec je skutečně možný," řekl Bento. O.K. "Takže alespoň matematicky to lze udělat."

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Reference

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If time had no beginning

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If time had no beginning

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Abstract

General Relativity traces the evolution of our Universe back to a Big Bang singularity. To probe physics before the singularity—if indeed there is a “before”—we must turn to quantum gravity. The Causal Set approach to quantum gravity provides us with a causal structure in the absence of the continuum, thus allowing us to go beyond the Big Bang and consider cosmologies in which time has no beginning. But is a time with no beginning in contradiction with a passage of time? In the Causal Set approach, the passage of time is captured by a process of spacetime growth. We describe how to adapt this process for causal sets in which time has no beginning and discuss the consequences for the nature of time.

1 Time and Causal Sets

Did time ever begin? It is hard to decide which answer is more unsettling: the idea of an infinite past with no beginning or the concept of such a beginning—the birth of the Universe. Stephen Hawking proved that General Relativity (GR) breaks down at a Big Bang singularity, but left open the possibility that the Big Bang is not the beginning of time but rather that it was preceded by a quantum gravity era which cannot be captured by GR [1]. The question of the beginning of time must therefore be addressed within a theory of quantum gravity.

Causal Set Theory is an approach to quantum gravity which postulates that spacetime is fundamentally discrete and takes the form of a causal set, a partial order whose elements

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are the indivisible "atoms" of spacetime [2,3]. The partial order is interpreted as a temporal order, so that the past of an element is formed of all the elements which precede it in the partial order. Thus the causal set furnishes a causal structure—a notion of before and after—in the absence of the continuum, allowing us to contemplate whether there was anything "before" the Big Bang (Fig.1) [4].

Figure 1: A causal set. Elements are represented as nodes and the order is indicated by the edges: element xprecedes element yif and only if there is an upward-going path from xto y. The portion of the causal set which lies in the shaded region is well approximated by a continuum spacetime (physics in this region is captured by GR). The remainder of the causal set forms the quantum gravity era preceding the Big Bang singularity.

Naively, we may consider the continuum spacetime of GR to emerge from an underlying causal set via a large (length) scale approximation [5]. But quantum mechanics suggests that reality is better described as a superposition of causal sets. A quantum theory of causal sets will ultimately be formulated as a sum-over-histories—a "path integral" of sorts—with the causal set playing the role of "history" or "spacetime configuration" [6–8]. Assigning a weight to each history in the sum is the problem of causal set dynamics.

Much of the effort towards obtaining a dynamics for causal sets has been guided by the paradigm of growth dynamics which states that the weight/action emerges from a fundamental physical process in which the causal set comes into being ex nihilo. This notion of becoming, the idea that a causal set grows element by element, further allows the passage of time to be captured by physics: an instantaneous moment—a now—corresponds to the birth (not to the existence) of an element [9–11].

Kinematically, causal sets can provide a cosmology in which time has no beginning—namely, a causal set in which every element has an infinite past. But are such past-infinite causal sets compatible with the heuristic of growth and becoming? If not, we may be forced

to choose between a passage of time and a beginningless time.

2 Growth Dynamics: Sequential vs Covariant

In its fully-fledged form, the growth process will be a quantum phenomenon [12–15] but at this stage of development of Causal Set Theory, growth dynamics are classical stochastic processes which generate infinite causal sets. Thus far, the most fruitful growth dynamics are the Classical Sequential Growth (CSG) models [16] in which, starting from the empty set, a single element is born at each stage (Fig.2). The ordering of each new-born element with respect to the already-existing elements is determined probabilistically according to each model but always satisfies the constraint that a new-born element cannot precede an already-existing one, ensuring a consistency between the interpretation of the partial order as a temporal order and of the birth of elements as the passage of time.

Figure 2: Sequential growth. Elements are born in a total order, one after the other. The total order of births is unphysical (pure gauge).

Our individual experience of the passage of time as a linear, totally ordered sequence of events is reflected in the sequential nature of the CSG models where elements are born in a sequence, one after the other. But this familiar notion of becoming is too simplistic to capture the intrinsic partial order/causal structure, since the total order acts as a gauge global time. The struggle between the gauge formulation of sequential growth and the gauge-independent nature of the physical world (cf. local coordinates and general covariance in GR) is resolved by identifying gauge-independent observables. The role of observables is played by stems, finite “portions” of a causal set which contain their own past (Fig.3). In other words, in CSG models the growing causal set is fully determined by its stems [17–19]. The CSG models are toy models of quantum cosmology but their original formulation shies away from the question at hand—whether time began—since the condition which prohibits new-born elements from preceding already-existing ones means that the growth process can only produce causal sets in which time has a beginning. Loosening this restriction by allowing new-born elements to precede already-existing ones opens a new avenue for causal set cosmology in which the problem of the beginning of time can be formalised [20]. But how should this new form of growth, in which the order of births is incompatible with the partial order, be understood? If element x precedes element y in the temporal partial order, what could it possibly mean for y to be born before x? It is hard to see how the growth can be

(a) (b)

Figure 3: (a) Stems and convex sets. The green “portion” is a stem because it is finite and it contains its own past (i.e. the past of each of its elements). The red “portion” is not a stem because it does not contain its entire past (e.g. it does not contain the green elements), but it is a convex set because it contains all the elements which lie in between its elements

in the partial order. The black “portion” is neither a stem nor a convex set. (b) Continuum analogues of stems and convex sets. A stem corresponds to any union of past lightcones whose total spacetime volume is finite. A causal set with no beginning contains no stems, just like a geodesically complete spacetime contains no past lightcone of finite spacetime volume. A convex set is a generalisation of the intersection of a past lightcone with a future lightcone.

considered a real physical process in this modified framework. Is a time with no beginning inherently incompatible with the notion of becoming?

The missing piece that may reconcile a beginningless time with a physical growth process is to replace our intuitive notion of sequential becoming with asynchronous becoming where elements are born in a partial (not a total) order [9–11]. What does it mean for elements to be born in a partial order? Through the lens of our largely sequential experience, asynchronous becoming may sound more like a fantastical riddle than a description of physical reality. It is the role of mathematics to make sense of notions which lie beyond our everyday experience, and it may be that new mathematics is what is needed to better understand asynchronous becoming and its consequences for the nature of time.

Covariant growth is an alternative to sequential growth which may contain the seed of asynchronous becoming [21, 22]. In its original formulation, covariant growth only produces causal sets in which time has a beginning. Taking its cue from the CSG models, covariant growth assumes from the outset that a causal set spacetime is fully described by its stems (i.e. that causal sets which share all the same stems are physically equivalent). Thus, in contrast to sequential growth, covariant growth does not keep track of individual element births but only of the stems contained in the growing causal set. The growth process can be illustrated as a sequence of sets, where the nth set in the sequence contains all the causal sets which have cardinality nand are stems in the growing causal set (Fig.4). When the process runs to completion (in the $n\rightarrow\infty$ limit) all stems are determined, thus fully determining the causal set spacetime grown in the process.

Figure 4: Covariant growth. The growth process does not keep track of the birth of individual elements but rather of the stems in the growing causal set. The nth set in the sequence contains all the causal sets which have cardinality nand are stems in the growing causal set, so that after nsteps all the stems of cardinality ≤nare determined.

While the process of becoming is explicit in sequential growth, it is implicit or “vague” [23] in covariant growth (e.g. at any finite stage of the growth process, one cannot say which portion of the causal set has already come into being). But if there is a process of becoming which can be associated with covariant growth, then it may be that it is this quality of vagueness which embodies asynchronous becoming and thus allows us to reconcile the passage of time with a beginningless time in Causal Set Theory.

3 Causal sets with no beginning

Covariant growth can be modified to accommodate growth of causal sets in which time has no beginning. The key is identifying the observables pertaining to these causal sets. A causal set with no beginning contains no stems, since if a portion of the causal set contains its own past then it must contain infinitely many elements, while stems have finite cardinality by definition. Instead, the role of observables is played by convex sets, “portions” of a causal set which, whenever they contain a pair of elements xand y, contain all elements which lie between xand yin the partial order (Fig.3). If finite convex sets encode all that is physical in a causal set, then we can adapt the covariant growth process for past-infinite causal sets simply by replacing stems with convex sets [20]. This new formulation of covariant growth keeps track of convex sets contained in the growing causal set. At stage n, all convex sets of cardinality nare fixed so that in the n→ ∞ limit the causal set spacetime is fully determined.

The significance of this new covariant formalism is twofold. First, this process is capable of growing all kinds of causal sets: in some time begins, in others it does not. Thus, whether time has a beginning or not is no longer a choice hardwired into our construction but rather a question which we can ask of the dynamics. Second, the implicit nature of the growth means that there is no immediate contradiction between the process of becoming and the past-infinite nature of a growing causal set. It will be up to future work to decide whether covariant growth can really be interpreted as a physical growth of past-infinite causal sets; whether there is a yet unknown formalism which better encompasses asynchronous becoming and in doing so captures the passage of a beginningless time; or whether the physics of passage dictates that time must have a beginning.

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f time had no beginning: growth dynamics

for past-infinite causal sets

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Abstract

We explore whether the growth dynamics paradigm of causal set theory is compatible with past-in□nite causal sets. We modify the classical sequential growth dynamics of Rideout and Sorkin to accommodate growth ‘into the past’ and discuss what form physical constraints such as causality could take in this new framework. We propose convex-subordersas the ‘observables’ or ‘physical properties’ in a theory in which causal sets can be past-in□nite and use this proposal to construct a manifestly covariant framework for dynamical models of growth for past-in□nite causal sets.

Keywords: quantum gravity, general covariance, time

(Some □gures may appear in colour only in the online journal)

1. Introduction

Much of the effort directed towards obtaining a dynamics for causal set theory has been guided by the paradigm of growth in which a causal set grows via a stochastic process of accretion of spacetime atoms.4After the pioneering work by Rideoutand Sorkin [6], work has concentrated on the classical domain—e.g. [7–11] —though work has also been done on investigating how quantum growth models might be constructed [11–15].

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4The other main avenue is to construct a ‘quantum state sum’ over causal sets each weighted by an amplitude, for example [1–5].

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The archetypal and to date most fruitful and most studied growth dynamics for causal sets is Rideout and Sorkin's family of classical sequential growth (CSG) models [6]. In each of these models, a single element is born at each stage and an infinite random causal set is grown when the process is run to infinity. A CSG model is constrainedby the requirement of 'internal temporality' namely that that at each stage of the process, the new element cannot be born to the past of—cannot precede in the causal set order—an element born at an earlier stage. This internal temporality constraint on the process fixes the sample space of the CSG model: it is the set of infinite past-finite causal sets, where the term past-finite will be precisely defined shortly. Essentially, the causal set universe grown in a CSG model must have a beginning, by definition of the model. As such, the CSG models rule out the possibility that there might, for example, have been an infinite sequence of epochs in a bouncing scenario, punctuated by infinitely many 'Big Crunch-and-then-Big Bang' events, prior to our present epoch.

In this work, we consider whether the growth dynamics paradigm necessarily entails past-finiteness or whether it can be compatible with past-infinite causal set cosmologies as suggested by Wüthrich and Callender [16]. In particular,we will investigate causal set cosmologies which are both past-infinite and future-infinite, i.e. cosmologies in which time has neither a beginning nor an end. After setting out notation and concepts in section 2, in section 3we modify the CSG models to accommodate growth of such causal sets. Already at this point, conceptual challenges arise, as might be anticipated. Perhaps the most pressing of these is that our new framework requires that new elements be born to the past of existing ones, thus making it (nearly if not entirely) impossible to conceive of the growth process as a physical process of becoming [17,18]. Nevertheless, we are able to identify a set of meaningful, comprehensible observables[5] for past-infinite growth dynamics, namely the convex-events that specify which convex-suborders are contained in the growing causal set. This sets the stage for section 4 where we pursue an alternative route to past-infinite growth by constructing a variation of covtree which is the basis of a manifestly covariant alternative to the framework of sequential growth models [19]. We show that the resulting framework is compatible with past-infinite growth and that the observables in this case are exactly the formerly identified convex-events. We conclude with a discussion in section 5.

2. Preliminaries

In this section we present terminology and notation that we use in the rest of this work, beginning with some standard terminology.

Let Π be a countable (finite or infinite) causal set (or 'causet' for short). We adopt the irreflexive convention for the relation on Π: $x \nprec x, x \in \Pi$. Recall that a causal set is locally finite by definition: $|\{z|x \prec z \prec y\}| < \infty \, \forall x, y \in \Pi$ such that $x \prec y$.

The past of $x \in \Pi$ is the subcauset $past(x) := \{y \in \Pi | y \prec x\}$. This is the non-inclusive past, i.e. $x \notin past(x)$. The future of $x \in \Pi$ is the subcauset $future(x) := \{y \in \Pi | y \succ x\}$. This is the non-inclusive future.

Π is past-finite if $|past(x)| < \infty \, \forall x \in \Pi$. Similarly, Π is future-finite if $|future(x)| < \infty \, \forall x \in \Pi$.

Π is past-infinite (future-infinite) if it is not past-finite (future-finite).

Π is two-way infinite if it is both past-infinite and future-infinite. Building growth dynamics for two-way infinite causet cosmologies is the motivation for this current work.

5 We use the term 'observable' as a shorthand for 'physical property' and not to imply that there need be any external observer.

A stem in Π is a finite subcauset Φ of Π such that if $x \in \Phi$ then $past(x) \subseteq \Phi$. An n-stem is a stem with cardinality n.

If Π is past-finite then an element $x \in \Pi$ is in level L in Π if the longest chain of which x is the maximal element has cardinality L, e.g. level 1 comprises the minimal elements of Π.

The width of Π, $w(\Pi)$, is the largest cardinality of an antichain in Π. The height of Π, $h(\Pi)$, is largest cardinality of a chain in Π. If Π is past finite, the height of Π equals the number of levels in Π. Note, the height and width may be infinite if Π is infinite.

A path in Π is a (finite or infinite) chain in Π such that the relation between each adjacent pair of elements in the chain is a link (i.e. a covering relation) in Π.

2.1. Natural labelings and labeled causets

Labeled causets as defined below are used throughout this paper. We emphasise that the definition of labeled causets which we give here is different to that given in [19]—it is an extension that allows us to discuss past-infinite causets. Correspondingly, definitions deriving from labeled causets (e.g. the definition of an n-order) and the symbols we use to denote spaces of labeled causal sets (e.g. $\tilde{\Omega}(n)$ and $\tilde{\Omega}$) take a different meaning here to that in [10,19].

Let Ψ be a countably infinite causet. Let $\mathbb{Z}^-$ be the set of negative integers.

A natural labeling of Ψ is a bijection f from either $\mathbb{N}$ or $\mathbb{Z}^-$ or $\mathbb{Z}$ to Ψ that satisfies

$f(i) \prec f(j) \implies i < j$.

The following lemma will be useful:

Lemma 2.1. Let Ψ be a countably infinite causet. Then,

(a) Ψ has a natural labeling by $\mathbb{N}$ if and only if Ψ is past-finite [20];

(b) Ψ has a natural labeling by $\mathbb{Z}^-$ if and only if Ψ is future-finite (a corollary of (a));

(c) Ψ has a natural labeling by $\mathbb{Z}$ if and only if one of the following conditions holds [21,22]:

(1) Ψ is two-way infinite;

(2) Ψ is past-finite and has infinitely many minimal elements;

(3) Ψ is future-finite and has infinitely many maximal elements.

Note that cases (c)(2) and (c)(3) are each disjoint from (c)(1) but not from each other, e.g. the in□nite antichain satis□es (c)(2) and (c)(3).

For any pair of integers k□l,let[k,l] denote the set of integers {k,k+1, ...,l−1, l}.Let Πnbe a □nite causet of cardinality n.

Anatural labeling of Πnis a bijection f:[k,k+n−1] →Πthat satis□es f(i)≺f(j)=⇒ i<j∀i,j∈[k,k+n−1], where k∈Z.

A□nite labeled causet is a causet with ground-set [k,l], where k□l, whose order satis□es the condition: x≺y=⇒x<y, i.e. it is a causet for which the identity map is a natural labeling (hence its name).

An in□nite labeled causet is a causet with ground-set Nor Z−or Zwhose order satis□es the condition: x≺y=⇒x<y(as in the □nite case, it is a causet for which the identity map is a natural labeling).

From now on we will denote labeled causets and their subcausets by capital Roman letters with a tilde, e.g. ˜

C. We often (but not always) use a subscript to denote the cardinality of a labeled causet, e.g. ‘˜

Cnhas cardinality n’.

Given some n∈N+, we denote the set of all labeled causets with cardinality nby ˜

Ω(n).

Note that given a labeled causal set ˜

Cn∈˜

Ω(n) with ground set [0, n−1], for each integer k

there is an isomorphic labeled causet with ground set [k,n−1+k] that is gotten from ˜

Cnby

adding k to each of its elements. Therefore there are infinitely many labeled causets in $\tilde{\Omega}(n)$.

This is not the case in previous works on past-finite causet growth models where the ground set of a finite labeled causet of cardinality n is fixed to be $[0, n-1]$.

The set of all infinite labeled causets whose ground set is $\mathbb{N}, \mathbb{Z}^-$ or $\mathbb{Z}$ respectively is denoted by $\tilde{\Omega}_{\mathbb{N}}, \tilde{\Omega}_{\mathbb{Z}^-}$ or $\tilde{\Omega}_{\mathbb{Z}}$, respectively.[6]

The set of all infinite labeled causets is denoted by $\tilde{\Omega} \equiv \tilde{\Omega}_{\mathbb{N}} \sqcup \tilde{\Omega}_{\mathbb{Z}^-} \sqcup \tilde{\Omega}_{\mathbb{Z}}$.

A CSG model [6] grows past-finite causal sets, i.e. its sample space is $\tilde{\Omega}_{\mathbb{N}}$.

2.2. Orders

We write $\tilde{C} \cong \tilde{D}$ if labeled causets $\tilde{C}$ and $\tilde{D}$ are equal up to an order-isomorphism.

An order, C, is an order-isomorphism class of labeled causets. We denote orders by capital Roman letters without a tilde.

Given an order C, its cardinality $|C|$ is defined to be the cardinality of a representative of C. Similarly, the width and height of an order are those of its representatives. An order is future-nite if its representatives are future-nite etc. When we refer to elements of C, we mean elements of a representative of Cand the meaning should be clear from the context as in for example: ‘Chas 5 minimal elements.’

An n-order is an order with cardinality n.

For each n∈N,Ω(n) denotes the set of n-orders. Ω(n) is a nite set.

Ω:=Ω/∼

=is the set of innite orders.

ΩZ,ΩZ−and ΩNare the subsets of Ωthat have a representative labeled by Z,Z−and N, respectively.

Note that Ω=Ω

Z∪ΩZ−∪ΩN. By lemma 2.1, the union is not disjoint. ΩZ∩ΩZ−∩ΩN=

ΩZ−∩ΩNis the set of past-and-future-nite orders that have innitely many maximal elements

and innitely many minimal elements and is nonempty: the union of innitely many disjoint two-chains for example. ΩZ∩ΩZ−is the set of future-nite orders that have innitely many maximal elements. ΩZ∩ΩNis the set of past-nite orders that have innitely many minimal elements.

2.3. Convex-suborders

Let Πand Ψbe causal sets.

Πis a convex-subcauset in Ψif Πis nite and Π⊆Ψand, whenever x,y∈Πand

x≺z≺yin Ψ,thenz∈Π.

We say that Ψcontains a copy of Πif there exists a convex-subcauset Π⊆Ψthat is order-isomorphic to Π.

Let Cand Dbe orders with (arbitrary) representatives ˜

Cand ˜

D, respectively.

We say that Cis a convex-suborderin Dif ˜

Dcontains a copy of ˜

C. Note that this denition is

independent of the representatives ˜

Cand ˜

Dbecause the de□nition of ‘contains a copy of’ is less

restrictive than ‘contains as a subcauset’. In that case we also say that Cis a convex-suborder

in ˜

D. If the cardinality of convex-suborder Cequals nwe say that Cis an n-convex-suborder

in Dor in ˜

D.

WesaythatanorderCis a convex-rogue if there exists another order Dthat is not isomor-

phic to Cand that has the same convex-suborders as C. In that case we say that Cand Dare a

convex-rogue pair.7

6A note of caution: in previous works on past-□nite causet growth models, the notation ˜

ΩNhas been used for the set

of all □nite labeled causal sets.

7This terminology follows that of [10] in which a pair of rogues are two past-□nite, non-isomorphic orders with the

same stems.

Note that the convex-subcausets (convex-suborders) are ordered by inclusion. If Ais a

convex-subcauset of Band Bis a convex-subcauset of C,thenAis a convex-subcauset

of C.

3. Sequential growth

The paradigm of growth dynamics is motivated by the heuristic concept of becoming: the dis-

crete causal set spacetime comes into being ex nihilo via an unceasing process of the birth of

causal set elements. While the concept of becoming could be regarded simply as a crutch in

de□ning a model of random in□nite causal sets and dispensed with once a measure has been de□ned on the full sigma algebra of events, Sorkin has proposed that growth is a physical pro-cess in which the birth of an element is the happening of an event, while the element itself signi□es that the event (of its birth) has already happened [17,18]. This viewpoint allows the passage of time to be manifested within physics as the growth of a causal set.

Perhaps the most intuitive notion of growth is that of ‘sequential growth’ in which the causal set grows through a sequential accretion of elements, somewhat akin to a tree growing at the tips of its branches. A sequential growth process for causal sets is made up of stages, labeled by the natural numbers, a discrete parameter. Starting at stage 0, at each stage nin the sequence a new element is born. The new element is born with randomly chosen relations with the already existing elements according to a model-dependent probability distribution. So, at the end of stage n, the growing, partial causet contains n+1 elements. In the limit $n\to\infty$, the process generates an in□nite causal set.

The CSG models are the archetype of sequential growth models. First introduced in [6], the CSG models have proved to be a fruitful arena for studying causal set cosmology [10,23–26] and for developing new dynamical frameworks [15,19,27]. Though the CSG models themselves do not generate past-in□nite causal sets, they are a natural starting point for trying to construct dynamics for two-way in□nite causal sets.

3.1. Alternating growth

As mentioned, the CSG models themselves do not generate past-in□nite causal sets. This is not a probabilistic statement: there are no past-in□nite causets at all in the sample space for the process. Each CSG model satis□es a condition known as internal temporality which states that at each stage the new element cannot be born to the past of—cannot precede in the causet order—an existing element. Indeed, the □rst challenge in generalising the CSG models to the past-in□nite case is generalising the condition of internal temporality. If we are to both generate past-in□nite causal sets and keep the essence of sequential growth—i.e. that starting from the empty set, new elements are born in a sequence of stages—we must loosen the condition of internal temporality to allow elements to be born to the past of existing elements. This move breaks the compatibility between the label of the stage of the sequential growth process, the concept of the birth of the element as manifesting the physical happening of the event and the order of the resulting causet as being the physical order—before and after—in which the elements are born.8Nevertheless, mathematically at least, there is a way to generalise the condition of internal temporality that keeps some of its power.

In a CSG model, Nis the ground set of the growing causal set, and at the stage labeled n the element nis born. In this context, internal temporality is equivalent to the requirement that

8Note that in [16] it is suggested that one could interpret this modi□cation physically as a ‘world [which] becomes in
both directions’, although we do not take this interpretation here.

Figure 1. The □rst three levels of labeled poscau.

the growing causal set is naturally labeled by N—i.e. the sample space of the growth process is ˜
ΩNand the growth process can be conceived of as a random walk up ‘labeled poscau.’9

Definition 3.1. Labeled poscau is the partial order on the set of □nite labeled causets whose ground set is [0, n], for all n∈N,where˜
S≺˜

$\tilde{R}$ if and only if $\tilde{S}$ is a stem in $\tilde{R}$.10

Labeled poscau is a rooted directed tree and its first three levels are shown in figure 1.

Reformulating internal temporality as a statement about natural labelings reveals a candidate generalisation of it to the two-way infinite case, namely that the infinite causal set that is grown has a natural labeling by $\mathbb{Z}$: it is an element of $\tilde{\Omega}_{\mathbb{Z}}$. Some freedom remains in how to translate this condition back into a statement about the sequence of the birth of the elements of the causet.

For definiteness, in this work we fix the freedom thus: let the positive and negative integers be born in an alternating sequence, $0, -1, 1, -2, 2 \ldots$, so that at stage n, if n is even the element $\frac{n}{2}$ is born and if n is odd the element $-\frac{n+1}{2}$ is born. We call a transition in which a positive element is born a 'forward transition'. Similarly, a 'backward transition' is one in which a negative element is born, so a transition $\tilde{C}_n \to \tilde{C}_{n+1}$ is forward when n is even and backward when n is odd. Note that, in this framework of alternating growth, the natural number label of the stage is not equal to the element of the causet born at that stage (as it is in CSG models) though it is still the case that the label of the stage equals the cardinality of the partial causet at the beginning of the stage (as it is in CSG models).

Internal temporality in this context becomes the condition: positive elements cannot be born to the past of elements born at previous stages, negative elements cannot be born to the future of elements born at previous stages. In particular, at no stage can an element be born between two elements that were born at previous stages. This implies that at each stage, the finite partial causet is a convex-subcauset of the partial causet at the next stage, and thence of the infinite causet that is the union of all the partial causal sets at all the infinitely many stages.

9‘Poscau’ is short for the ‘partial order of causal sets’, and ‘labeled’ signiﬁes that the causal sets in the order are

labeled causets.

10 We use the symbol $\prec$ to denote the relation for several different partial orders in this work. The meaning of $\prec$ in

each case is to be inferred from the context.

Figure 2. The ﬁrst three levels of alternating poscau.

We dub the resulting dynamical framework ‘alternating growth’. The alternating growth process can be represented as a random walk up ‘alternating poscau’, a directed rooted tree whose nodes are ﬁnite labeled causets. More precisely,

Definition 3.2. Alternating poscau is the partial order on the set of ﬁnite labeled causets whose ground set is $[-n,n]$ or $[-n,n+1]$ for all $n>0$, where $\tilde{S} \prec \tilde{R}$ if and only if $\tilde{S}$ is a convex-subcauset in $\tilde{R}$.

The ﬁrst three levels of alternating poscau are shown in ﬁgure 2.11 Note that the levels of alternating poscau are ﬁnite because of the restriction on the ground sets of the ﬁnite labeled causets to $[-n,n]$ or $[-n,n+1]$.

There is a bijection from the set of inﬁnite paths starting at the root in alternating poscau to $\tilde{\Omega}_{\mathbb{Z}}$, where an inﬁnite path $\tilde{C}_1 \prec \tilde{C}_2 \prec \ldots$ maps to $\tilde{}$

$C = \bigcup_{n>0} \tilde{C}_n$. The standard technology of stochastic processes and measure theory then provides the σ-algebra of measurable events generated by the semi-ring of all cylinder sets, each associated with a node of alternating poscau: $cyl(\tilde{C}_n) \subset \tilde{\Omega}_Z$ is the set of labeled causets on ground set Z that contain $\tilde{C}_n$ as a convex-subcauset. A random walk on alternating poscau specified in terms of transition probabilities corresponds to a unique measure on this measurable space and, vice versa, every measure on the σ-algebra generated by the cylinder sets gives a unique collection of transition probabilities for every transition.

By re-interpreting the internal temporality condition as above, we are thus able to modify the sequential growth paradigm to allow growth of two-way infinite causets.[12] Recall, however, that by lemma 2.1 the set of two-way infinite causets (case (c)(1) in lemma 2.1) is a proper subset of the sample space $\tilde{\Omega}_Z$. It turns out that the set of two-way infinite causets is a measureable set and therefore it will be up to the dynamics (i.e. the specific random walk) whether the set of two-way infinite causets has measure one or not. Indeed, one can ask whether one can identify

11 There is a bijection, f from the set of nodes at level k in labeled poscau to the set of nodes at level k in alternating poscau where f takes a labeled causet and maps each element x to $x - \lfloor k/2 \rfloor$. Note however that f is not an isomorphism between labeled poscau and alternating poscau.

12 One can consider growth models with different rules, leading to different trees: we refer the reader to [28] for such variations, e.g. sequential growth models in which the decision to make a forward or a backward transition at each stage is random.

conditions on the transition probabilities that will imply that the causet will almost surely be two-way infinite.

Lemma 3.3. Let W be the set of two-way infinite labeled causets. W is a measureable set in an alternating growth d ynamics, i.e. a random walk up alternating poscau.

Proof. $W = W^+ \cap W^-$ where W^+ (W^-) is the set of causets in $\tilde{\Omega}_{\mathbb{Z}}$ that have an element with an infinite future (past). We will show that W^+ is measureable and the proof for W^- is similar. For each integer $k \in \mathbb{Z}$ let Γ_k be the set of causets in $\tilde{\Omega}_{\mathbb{Z}}$ such that the element k has an infinite future. W^+ is the union of all the Γ_k.

For each $k \in \mathbb{Z}, m, n \in \mathbb{N}$ s.t. $m > 0$ and $n > |k| + m$ let $\tilde{\Omega}_{k,n,m}$ be the set of finite labeled causets on the ground set $[-n, n]$ such that there are m elements above element k. Take the union over the set $\tilde{\Omega}_{k,n,m}$ of all the associated cylinder sets and call that union $\Gamma_{k,n,m}$:

$$\Gamma_{k,n,m} := \bigcup_{\tilde{C} \in \tilde{\Omega}_{k,n,m}} \mathrm{cyl}(\tilde{C}). \quad (1)$$

Then

$\Gamma_k=$

∞

□

$m=1$

∞

□

$n=|k|+m+1$

$\Gamma_{k,n,m}$.(2)

□

3.2. Alternating growth dynamics

While any random walk on alternating poscau gives rise to a well-de□ned measure space with sample space ˜

Ω_Z, not every such walk will be interesting physically and it remains for us to identify classes of interest.

This is completely analogous to the past-□nite case, where the CSG models were identi□ed as a physically-meaningful subclass of the random walks on labeled poscau. Indeed, the CSG models are exactly the random walks on labeled poscau that satisfy the physically motivated conditions of discrete general covariance (DGC), and Bell causality, to be discussed further below. These conditions were solved and the transition probabilities in a CSG model proved to take the following form:

P(˜

$C_n\rightarrow$˜

$C_{n+1})=\lambda($□$,m)$

$\lambda(n,0)$,(3)

where P(˜

$C_n\rightarrow$˜

C_{n+1}) is the probability of transition from ˜

$C_n$to one of its children, ˜

C_{n+1};□

and mare the number of new relations and new links, respectively, formed with the newborn element at stage n; and the function λis given by,

λ(k,p):=

k−p

□

i=0□k−p

i□tp+i,(4)

where {t0,t1,t2,...}is an in□nite set of real non-negative parameters or ‘couplings’ (with t0>0) that specify the particular CSG model. As the transition probabilities are ratios of linear combinations of the tn’s, there is a (projective) equivalence relation on the sets {tn}, which freedom can be □xed by setting t0=1.

Note that abovewe referred tothe ‘new relations andnew links [...] formed withthe new-born element at stage n’ without mentioning that the newborn element is nin a CSG model and without mentioning that in any new relation the newborn element must succeed (be above) the element born at a previous stage. We have made these omissions because, by doing so, we can adopt equation (3) and its succeeding text de□nition as is for the de□nition of an alternating growth model simply by letting ˜

Cnand ˜

Cn+1denote nodes in alternating poscau such that

˜

Cn→˜

Cn+1is a possible transition in alternating poscau. Now, however, when the stage label n is odd the transition is a backward transition and in any new relation the newborn element must precede (be below) the already existing element. We call this new family of models the family of alternating CSG dynamics [21,22]. Given a CSG model with parameters {t0,t1,t2,...}, its alternating counterpart is the alternating CSG model with the same set of parameters.

Do the alternating CSG dynamics retain any of the features that make CSG models physically interesting? For example, do the Alternating CSG models satisfy any sort of causality condition? In the remainder of this section we identify the form that four key attributes—covariance, causality, causal immortality and meaningful observables—might take in the alternating growth framework and discuss whether the alternating CSG models possess these attributes.

Before turning to the question of physical conditions, we introduce the example of the most well-studied family of CSG models, transitive percolation (TP)—a one-parameter family of CSG models given by $t_k=t^k$ where t is a positive real constant [6,29]. Its alternating growth counterpart, alternating TP, is defined by the same couplings: $t_k=t^k$ for some $t>0$. For TP, the transition probability given in equation (3) takes the simple form,

$$P(\tilde{C}_n \to \tilde{C}_{n+1}) = p^m q^{n-\ell}, \quad (5)$$

where $p=\frac{t}{1+t}$ and $q=1-p$, so that p□=1 and p□=0 (and as before, ℓ and m are the numbers of new relations and new links, respectively, formed with the element that is born at stage n).

The interpretation of equation (5) is that the new element born in the transition forms a relation with each existing element with probability p independently and then the transitive closure is taken to obtain $\tilde{C}_{n+1}$. With this interpretation, t^k is the relative probability that the new element forms exactly k relations (before taking the transitive closure). Equation (5) and the functional form of the couplings $t_k=t^k$ reflect the ‘local’ nature of TP. All other CSG models can be seen as ‘non-local’ generalisations of TP in which the probability of the newborn forming a relation with a given element depends on whether or not relations are formed with the other existing elements.

Equation (5) and hence this form of ‘locality’ is retained by alternating TP, so that there are close similarities between the two models. For example, let $\tilde{C}_n$ and $\tilde{D}_n$ be nodes in labeled poscau and alternating poscau respectively, and let $\tilde{C}_n \cong \tilde{D}_n$. Then [21,22],

Lemma 3.4. The probability of reaching $\tilde{C}_n$ in a particular CSG dynamics is equal to the probability of reaching $\tilde{D}_n$ in the alternating growth counterpart of that CSG dynamics if and only if the CSG dynamics in question is TP.

Thus, at any finite stage of growth, TP and alternating transitive percolation (ATP) cannot be distinguished. However the processes are different when run to infinity. For example, in TP, the past finite causet grown almost surely has infinitely many posts i.e. infinitely many elements $\{k_1,k_2,...\}$ such that $0 \prec k_1<k_2<k_3...$ and every element of the causet is related to all the k_i. In ATP, almost surely a two-way infinite causet is grown in which there are again infinitely many posts in the past and in the future: elements $\{...k_{-2},k_{-1},k_0,k_1,k_2...\}$ such that $...k_{-2}<k_{-1}<k_0<k_1<k_2...$ and such that every element of the causet is related to all

the ki. TP realises the heuristic of a bouncing universe with a beginning and ATP realises the heuristic of a bouncing universe with no beginning.

Covariance: it is a tenet of causal set theory that the atoms of spacetime have no structure; it is of no physical relevance what mathematical objects the elements of a causal set are.13 Only the cardinality of the causet and the order relation are physical. This implies that the mathematical identity of and labels of the causet elements are not physical and one can consider this an analogue of the ‘coordinate invariance’ or ‘general covariance’ of continuum general relativity.

In a CSG model of past-finite sequential growth, this label invariance is manifested thus: given a pair of order-isomorphic finite labeled causets, $\tilde{C}_n$ and $\tilde{C}'_n$, which are nodes in labeled poscau, the probability of reaching $\tilde{C}_n$ is equal to the probability of reaching $\tilde{C}'_n$, that is,

$$\tilde{C}_n \cong \tilde{C}'_n \implies P(\tilde{C}_n) = P(\tilde{C}'_n). \quad (6)$$

Condition (6) is known as DGC and it can be generalised to pertain to the alternating sequential growth framework simply by letting $\tilde{C}_n$ and $\tilde{C}'_n$

nin equation (6) denote isomorphic nodes in
alternating poscau.

Every CSG model satis□es the DGC condition. In contrast, the only alternating CSG dynamics which satis□es DGC is ATP:

Claim 3.5. An alternating CSG model satis□es the DGC condition if and only if it is an ATP model.

Proof. That ATP satis□es DGC follows from equation (5) since it implies that the probability of reaching some ˜
Cnin alternating poscau is P(˜
Cn)=pLq(n
2)−R,whereLand Rare the number
of links and relations in ˜
Cn, respectively. These numbers Land Rdepend only on the order-
isomorphism class of ˜
Cn.

Now consider an alternating CSG model de□ned by parameters {tn}. Consider, for n>0, the (2n+1)-order Cthat contains a (2n)-antichain of which nelements have a common ancestor, as shown in □gure 3.

Let ˜
Cdenote the representative of Cthat is grown in the alternating framework in the fol-lowing way: the element 0 and the elements born in the □rst 2n−2 stages form an antichain, the element born at stage 2n−1 is born to the past of nof the existing elements, and the ele-ment born at stage 2nis unrelated to all existing elements. The probability of growing ˜
Cin an
alternating CSG dynamics is P(˜
C)=t2n−2
0tnt0=t2n−1
0tn.Let ˜
C□denote another representative
of Cthat is grown in the alternating framework in the following way: the elements born in

forward transitions are all born to the future of the element 0, and the elements born in backward transitions are all born unrelated to all existing elements. The probability of growing ˜

C□

in an alternating CSG dynamics is P(˜

C□)=tn

1tn

0. If the alternating CSG model is covariant then

P(˜

C)=P(˜

C□), which implies that tn

1

tn=tn−1

0. This is ATP and can be cast into the form tn=tn

by setting t0=1. □

Causality: within the past-□nite sequential growth framework, a dynamics is causal if it satis□es the 'Bell causality' condition of [6] which adapts the 'local causality' condition of Bell's theorem to a causal structure that is discrete and dynamical. The Bell causality condition

13 Our choice of labeled causets —with their ground sets of integers —for our world of discourse in this paper is purely

for convenience.

Figure 3. The (2n+1)-order C, that contains a (2n)-antichain of which nelements have

a common ancestor.

Figure 4. An illustration of the Bell causality condition in past-□nite sequential growth.

The parent ˜

C7and two of its children are shown on the top line. From these, the parent

˜

B6and two of its children can be obtained by removing the element 4 (i.e. the spectator)

and relabeling. The new-born element in each child is shown in white. The past of the

new-born element in each transition is shown in red.

takes the form of an equality between ratios of transition probabilities,

P(˜

Cn→˜

Cn+1)

P(˜

Cn→˜

C□

n+1)=P(˜

Bl→˜

Bl+1)

P(˜

Bl→˜

B□

l+1),(7)

where ˜

Bl+1,˜

B□

l+1and ˜

Blare obtained from ˜

Cn+1,˜

C□

n+1and ˜

Cn, respectively, by deleting one or

more spectators14 and then relabeling the remaining elements consistently. One concrete way

to do this relabeling after deletion of spectators from ˜

Cnis to shift all the labels down, lling

in the gaps without changing the total order, as necessary until the ground set is [0, l−1]: this

is then ˜

Bl. An example is shown in gure 4. For a model that satises DGC, the algorithm

14 A spectator is an element that is spacelike to the newborn element in both transitions ˜

Cn→˜

Cn+1and ˜

Cn→˜

C

n+1.

Figure 5. The Bell causality condition is not always well-dened in the alternating

growth framework. The parent ˜

C4and two of its children are shown on the top line. The

new-born element in each child is shown in white. The past of the new-born element in

each transition is shown in red. ˜

B3is constructed from ˜

C4by removing the element 1

(i.e. the spectator) and relabeling. Removing the spectator from ˜

C5and ˜

C

5results in the

causets shown in the box, but there is no relabeling of these that corresponds to children of ˜

B3.

for consistent relabeling after deletion of spectators plays no real role because the transition

probabilities do not depend on the labeling, only the order-isomorphism class of the causets.

So we can say that (7) holds for all relabelings, but is only one independent condition when

the dynamics satis□es DGC.

What form can the Bell causality condition take within the alternating growth framework?

While at □rst glance it may seem that equation (7) can be adapted to the alternating framework

simply by letting ˜

Bl+1,˜

B□

l+1,˜

Bl,˜

Cn+1,˜

C□

n+1and ˜

Cndenote nodes in alternating poscau, this is

not so. To see this, let ˜

Cnbe a node in alternating poscau, and let ˜

Cn+1and ˜

C□

n+1denote two

of its children. Now, construct ˜

Blfrom ˜

Cnby removing the spectators and relabeling. Next,

remove the spectators from ˜

Cn+1and relabel—this is where the problem arises since there

may be no relabeling that produces a child of ˜

Bl. In particular, this failure occurs whenever

the number of spectators is odd because in that case if ˜

Cn→˜

Cn+1is a forward transition then

˜

Bl→˜

Bl+1must be a backward transition, which leads to a contradiction. An example is shown

in figure 5. It is in these cases that the generalisation of equation (7) to the alternating dynamics

becomes ill-defined. Instead, we will use a weakened causality condition (in similarity to the

weakened causality conditions of [25,30]) that states that an alternating dynamics is causal if

equation (7) is satisfied whenever there is a relabeling such that the condition is well-defined.

Having arrived at a proposed Bell causality condition for the alternating framework, we

can ask whether the alternating CSG dynamics satisfy it, beginning with ATP. Since ATP is

covariant (as we showed in claim 3.5), the relabeling issue in the Bell causality condition is

moot and we can use equation (5) to verify that equality (7) is satisfied,

P(˜

Cn→˜

Cn+1)

P(˜

Cn→˜

C

n+1)=pmqn−□

pm□qn−□□=pmql−□

pm□ql−□□=P(˜

Bl→˜

Bl+1)

P(˜

Bl→˜

B□

l+1),(8)

where □□and m□denote the number of relations and links, respectively, formed by the element that is born at stage nin the transition ˜

Cn→˜

C□

n+1. In this sense, the ATP models are

'Bell causal'. Since the remaining alternating CSG dynamics are not covariant, to ascertain whether they are causal either requires specifying a canonical method of relabeling by which ˜

Bl+1should be obtained from ˜

Cn+1etc, which renders the Bell causality condition itself label-dependent and hence not covariant, or the condition (7) should be imposed for each consistent relabeling that exists.

Having discussed formally adapting equality (7) to the alternating growth framework, we turn to the question of the physical interpretation of this proposed new Bell causality condition which is far from clear. The 'local causality' condition in Bell's theorem captures the heuristic that the outcome of a given event can only be in□uenced by the events inside its past lightcone. In this spirit, within the framework of past-□nite growth, the 'Bell causality' condition (equation (7)) states that at each stage of the growth process, the probability for each transition depends only on the past of the new-born element. But this interpretation is obliterated in the alternating growth framework. In a forward transition, the transition probability depends only on the past of the new-born element—but not on its entire past, since some of it has not yet been determined. The situation is even worse in the backward transitions where the transition probabilities depend on the future of the new-born element. One resolution is to require that equality (7) holds only for the forward transitions (i.e. when n is even), leaving the backward transitions unconstrained by causality. Or it may be that we need an altogether new way of thinking about causality in the alternating framework, if sense can be made of it at all.

Causal immortality: in past-finite sequential growth, the sample space is the space of all infinite past-finite causets, $\tilde{\Omega}_{\mathbb{N}}$. This space contains a variety of cosmologies: some are future-infinite and some are future-finite, some contain infinite antichains and some do not. But in the CSG models, only a subset of all these potential configurations can be realised because the CSG models generate, with probability one, causets with no maximal elements [10]. We say that the CSG models have the property of 'causal immortality' because the effect of each element/event reaches arbitrarily far into the future.

Similarly, in alternating sequential growth the sample space $\tilde{\Omega}_{\mathbb{Z}}$ contains several causal set families (given in lemma 2.1) but only a subset of these is realised by the alternating CSG dynamics because these dynamics generate causets with no maximal nor minimal elements, as we show in claim 3.6 below.

Claim 3.6. Every element in a causal set grown in an alternating CSG model with $t_k>0$ for some $k>0$ almost surely has an element to its future and an element to its past.

Proof. Consider a growth process with an alternating CSG dynamics with $t_k>0$ for some $k>0$. Suppose that the labeled causet $\tilde{C}_n$ has been grown by the beginning of stage $n>k$, and let $x\in\tilde{C}_n$ be a maximal element.

First, we show that the probability that xis maximal in the complete causal set is zero. Let r□nbe an even integer. Then the probability that xis maximal at the end of stage r(given that xis maximal at the beginning of stage r)is,

1−pr=λ(r−1, 0)

λ(r,0) .(9)

where pris the effective parameter of [31]. Therefore the probability that xis maximal in the complete causet is,

lim

s→∞ P(xis maximal at end of stage s)=lim

s→∞ □

even n□r□s

(1 −pr) (10)

which converges to a non-zero value if and only if the following series converges [32],

lim

s→∞

∞

□

even n□r□s

pr.(11)

Rearranging equation (9)wehave,

pr=□r

l=1□r−1

l−1□tl

λ(r,0) =1

r□□r

l=1

r!

(r−l)!(l−1)!tl

λ(r,0) □□1

r□□r

l=1□r

l□tl

t0+□r

l=1□r

l□tl□□1

r□tk

t0+tk□

(12)

and therefore the series (11) is divergent and probability (10)vanishes.

The argument can be adapted to show that every element has an element to its past by letting xbe a minimal element and letting rtake odd values. □

Observables: identifying the observables of quantum gravity is a challenge shared by all approaches. Within the sequential growth paradigm, the candidate observables are the measurable events that are covariant: a measurable event Eis covariant if ˜

C∈E =⇒˜

C□∈E

whenever ˜

C∼

=˜

C□. The challenge is to understand which of these candidate covariant events

have a comprehensible physical interpretation. A rich class of observables known as ‘stem-events’ has been identi□ed within the past-□nite sequential growth framework [9,10]. Each ‘stem-event’ corresponds to a logical combination of statements about which □nite orders are stems15 in the growing causet.

What are the analogous observables within the alternating growth framework? Stem-events are indeed measurable in the alternating framework:

Lemma 3.7. Stem-events are measureable in an alternating growth model.

Proof. First, we give a precise de□nition of stem-events within the alternating growth framework. For each □nite order Cnde□ne the set,

stem(Cn):={˜

D∈˜

ΩZ|Cnis a stem in ˜

D}.(13)

A stem-event is an element of the σ-algebra generated by the collection of the stem(Cn)’s. Now,

we show that each stem(Cn) can be constructed countably from the cylinder sets associated with

the nodes of alternating poscau and the result follows.

15 A □nite order Sis a stem in the order Cif there exists a representative of Sthat is a stem in some representative of

C. A □nite order Sis a stem in the labeled causet ˜

Cif the order Sis a stem in the order [˜

C][19].

Let $\tilde{C}_n$ be an arbitrary representative of C_n. Let $\tilde{\Omega}_{m,k}(C_n)$ denote the set of causets $\tilde{D}_k$ of cardinality k which are nodes in alternating poscau satisfying the following: $\tilde{D}_k$ contains a stem that does not contain any element outside of the interval $[-m,m]$ and that is isomorphic to $\tilde{C}_n$. Take the union over the set $\tilde{\Omega}_{m,k}(C_n)$ of the associated cylinder sets and call that union $\Gamma_{m,k}(C_n)$:

$$\Gamma_{m,k}(C_n) := \bigcup_{\tilde{D}_k \in \tilde{\Omega}_{m,k}(C_n)} \mathrm{cyl}(\tilde{D}_k). \quad (14)$$

Then take the intersection over k,

$$\Gamma_m(C_n) := \bigcap_k \Gamma_{m,k}(C_n), \quad (15)$$

and the union over m,

$$\mathrm{stem}(C_n) = \bigcup_m \Gamma_m(C_n). \quad (16)$$

□

But the freedom to grow past-infinite causal sets means that the stem-events have a weak distinguishing power—they tell us nothing about the past-infinite part of a casual set and they cannot distinguish between causets with no minimal elements which have no stems. We can make progress by noticing that stems are to past-finite growth what convex-suborders are to alternating growth. The ordering of labeled poscau is determined by the stem relation (i.e. the order of labeled poscau is order-by-inclusion-as-stem, cf definition 3.1), while the ordering of alternating poscau is order-by-inclusion-as-convex-subcauset(cf definition 3.2). Each node in labeled poscau is a stem in the growing causet, while each node in alternating poscau is a convex-subcauset in the growing causet. Therefore, we propose that 'convex-events' are the observables for alternating growth, as stem-events are for past-finite growth.

To make this precise, for each finite order C_n let $convex(C_n) \subset \tilde{\Omega}_{\mathbb{Z}}$ be the set of causets that contain C_n as a convex-suborder.

First we prove:

Lemma 3.8. For each finite order C_n, $convex(C_n)$ is measureable in an alternating growth model.

Proof. C_n is a convex-suborder in causet $\tilde{C} \in \tilde{\Omega}_{\mathbb{Z}}$ if and only if there exists a finite integer N such that C_n is a convex-suborder in the partial causet $\tilde{C}|_{[-N,N]}$ which is $\tilde{C}$ restricted to the interval $[-N,N]$.

For each $N \in \mathbb{N}$, let $\Gamma_N(C_n) := \bigcup_{\tilde{D}_N} cyl(\tilde{D}_N)$, where the union is over all labeled causets of cardinality N which are nodes in alternating poscau, $\tilde{D}_N$, such that C_n is a convex-suborder of $\tilde{D}_N$.

Then we have

convex(C_n)= $\bigcup$

N

ΓN(Cn).(17)

□

By de□nition, convex(Cn) is a covariant event and is therefore in the physical σ-algebra of covariant measureable events.

15

Figure 6. The ‘in□nite comb’ (left) and the in□nite comb disjoint union a single element (right) are convex-rogues since they contain the same convex-suborders as each other.

Let us also de□ne generally a ‘convex-event’ to be any event in the σ-algebra generated by all the convex(Cn)’s. Each convex-event is then a covariant measurable event with a clear physical meaning—it corresponds to a logical combination of statements about which □nite orders are convex-subordersin the growing causet. Causal intervals—Alexandrov sets—are important structures in the continuum. If Cnis an n-order with a single maximal element and a single minimal element then the convex-event convex(Cn) corresponds to the property ‘Cnis an order interval (somewhere) in the universe’. This corresponds to the continuum property: ‘the universe contains (somewhere) a causal interval with such-and-such geometry’.

Convex-events form a large class of observables which provide us with information about the structure of the causal set. But they cannot distinguish between pairs of ‘convex-rogues’, pairs of non order-isomorphic causal sets that have the same convex-suborders (an example is shown in □gure 6). In the past-□nite framework, the stem-events are also not fully-distinguishing since they fail to distinguish between pairs of ‘rogues’16.Howeveritwasshown in [10] that in any CSG dynamics the set of rogues has measure zero and therefore, in a precise sense, the stem-events exhaust the set of observables in any CSG dynamics. Crucially, the

result of [10] depends on the speci□cs of the CSG dynamics and does not hold for every random walk on labeled poscau but only for those models in which the set of rogues has measure zero.

Investigating the consequences of the claim that convex-events exhaust the comprehensible observables in an alternating CSG dynamics, we nd that in the only alternating CSG that satises DGC—namely ATP—the convex-events fail to provide any useful predictions. This is because in ATP (and TP) every nite order is almost surely a convex-suborderin the causet grown: i.e. the measure of every event convex(Cn) is equal to 1 [31]. If we dene a model to be ‘deterministic with respect to convex-events’ if every convex-event has measure zero or one, then ATP is deterministic with respect to the convex-events. Indeed, the causet grown will

almost surely contain innitely many copies of every convex-suborder: no matter where you are in an ATP universe, a copy of each nite order will occur in your future if you wait long enough just as a given nite bit string will almost surely occur innitely many times in an innite random string. The stem-events in TP, anchored as they are to the beginning, do not suffer from this problem. So convex-events cannot, with probability one, distinguish between

16 If a pair of non order-isomorphic causal sets, $\tilde{C}, \tilde{D} \in \tilde{\Omega}_N$, have the same stems as each other then each is called a ‘rogue’ and together they form a ‘rogue pair’ . If $\tilde{C}$ and $\tilde{D}$ are a rogue pair then every stem-event contains either both or neither.

any two innite causets grown in ATP—any two innite causets grown in ATP are almost surely a convex-rogue pair—and one can make no useful predictions using convex-events.

This example of ATP is important because TP holds a special position in the class of CSG models. It is a fixed point under the transformations known as cosmic renormalisation [23] that are the basis for the hope that causal set cosmology might be self-tuning and avoid the fine-tuning of cosmological parameters we find in our current standard cosmological model [33]. This failure means one must give up on ATP as a useful cosmological model or one must look harder for meaningful observables or one gives up on growth models that allow two-way infinite causets altogether.

In the rest of the paper, we take the first choice above: we adhere to the proposal of convex-events as the meaningful observables, accept that this means that ATP is not a useful cosmological model and explore models that allow two-way infinite causets in which there are non-trivial predictions about convex-events. First, we show that not every alternating CSG model is deterministic with respect to the convex-events:

Claim 3.9. An alternating CSG dynamics is not deterministic with respect to convex-events if its couplings are given by,

$$t_0=1 \text{ and } t_n=f(n)\lambda(n-1, 0) \quad \forall n \geq 1, \quad (18)$$

where $f(n)$ is a function satisfying $\sum_1^\infty \frac{1}{f(n)}<\infty$ (e.g. $f(n)=x^n$ with $x>1$ or $f(n)=n^s$ with $s>1$).

Proof. Let A_2 denote the two-antichain order, and let C_∞ denote the two-way infinite chain order. Note that $P(C_\infty)=1-P(\text{convex}(A_2))$, where $P(C_\infty)$ is the probability of growing C_∞ and $P(\text{convex}(A_2))$ is the measure of convex(A_2). By considering stage 1 of the growth we see that $P(\text{convex}(A_2)) > t_0/\lambda(1, 0) > 0$ in any alternating CSG dynamics. We will show that in the dynamics (18), $P(C_\infty)>0$ and therefore $0 < P(\text{convex}(A_2)) < 1$ and the result follows.

Now, $P(C_\infty)=\prod_{n>0} p_n$, where (as in claim 3.6) p_n is the effective parameter given by,

$p_n=\sum_{k=0}^{n-1}$

$\sum_{k=0}^{n-1}$

k□tk+1

λ(n,0) =λ(n,0)−λ(n−1, 0)

λ(n,0) , (19)

and the product converges to a non-zero value if and only if the series □(1 −pn)converges [32]. We can write the mth term of this series as,

1−pm=λ(m−1, 0)

λ(m,0) =□□m−1

r=0□m

r□tr

λ(m−1, 0) +tm

λ(m−1, 0)□−1

, (20)

and then substitute the couplings given in (18) to □nd,

1−pm=□□m−1

r=0□m

r□tr

λ(m−1, 0) +f(m)□−1

□1

f(m).(21)

It follows that in the models given in (18)thesum□(1 −pn) converges by the comparison test against □1

f(n)and hence P(C∞)>0. □

The existence of non-deterministic alternating growth models encourages us to continue to explore dynamics that allow two-way in□nite causets to grow. We might use the concept of convex-events to formulate constraints or guiding principles in searching for interesting alternating growth dynamics—e.g. a stronger condition than that the dynamics is not deterministic with respect to convex-events is that the dynamics almost surely does not generate convex-rogues. It is not known whether the dynamics (18) satis□es this condition.

In summary, in this section we generalised the sequential growth paradigm to accommodate two-way in□nite cosmologies. The resulting alternating framework generates causets that have a natural labeling by Z. We considered what form various physical conditions take in the alternating framework and whether an alternating generalisation of the CSG models satisfy them. Finally, we identi□ed the convex-events as a physical class of observables. The convex-events cannot discriminate between causets grown in an ATP model, which model is the only alternating CSG model that satis□es DGC. This means that if we want to demand DGC in an alternating growth model, that model cannot be an alternating CSG model.

In the next section we use the convex-events to provide an alternative to alternating sequential growth: a covariant framework for two-way in□nite growth, analogous to the existing covariant framework for the growth of past □nite causets [19,27].

4. Covariant growth

Sequential growth models are named for the way they are de□ned, with the causal set elements being born in a sequence of stages, with speci□ed transition probabilities for the possible transitions at each stage. This linear order of the stages is a gauge—a kind of supertime—since it is a tenet of causal set theory that only the partial order of the causet itself is physical. In other words, the de□nition of sequential growth models makes the elements of the growing causal set mathematically distinguishable or ‘labeled’—since elements are distinguished/labeled by the stage at which they are born—but some of this labeling information is unphysical since in causal set theory only the order-isomorphism class of the causet is physical. The dissonance between the labeled nature of sequential growth and the label-independent nature of the physical world finds a resolution once one has identified the covariant, label-independent observables and restricted oneself to making statements only about them. Thus, sequential growth models are a proof of concept for the growth dynamics paradigm and a playground in which to explore the dichotomy of being and becoming [17,18].

Covariant growth of past-finite causets is an alternative framework to sequential growth in which label independence is manifest from the start [19,27]. Its motivation is rooted in the notion of partially ordered growth or asynchronous becoming, in which the world comes into being—becomes—in a manner compatible with a lack of physical global time through a partially ordered process of the birth of spacetime atoms [17,18]. Covariant growth models seek to bypass the introduction of the unphysical gauge in sequential growth—the linear order of the stages at which the causet elementsare born one by one—and to deal only with covariant quantities throughout. This is an ambitious project and we anticipate that the struggle between the local nature of the dynamics of a gauge field and the global nature of gauge invariant quantities will play out in pursuing it.

Thus far, covariant growth has only been explored in the context of past-finite orders where the dynamics takes the form of a random walk up covtree, a partial order that is a directed tree whose nodes are sets of orders.[17] At level nof covtree, each node is a set of n-orders, interpreted as the set of n-stems of the growing past-finite causet. This interpretation is founded on the theorem that for each inextendiblepath up covtree there indeed exists an infinite order whose n-stems form the node in that path at level n[19]. This dynamics pertains to covariant properties

17 Recall that ‘order’ is short for ‘order-isomorphism class’ (see section 2.2).

of the causet from the outset and no labeling is introduced. The random walk progresses in stages from each level of covtree to the next. At the beginning of stage $n$of the random walk, the 'state' of the process is a node in level $n-1$—that is, the $(n-1)$-stems of the growing causet have already been chosen. At stage n, the walk transitions to a new node in level $(n-1)$: i.e. the set of n-stems of the growing causal set universe is chosen at random according to the transition probabilities of the model. And so on.

Note that in this scenario of covariant growth, the manifest label-independence comes at the 'cost' of the model not making direct reference to the process of the birth of individual space-time atoms: in a sequential growth model—i.e. a random walk up labeled poscau—element $n$is born at stage $n$and that is not the case in covariant growth on covtree. A covtree node at stage n,Γ_n, is a collection of n-orders and corresponds to the statement '$\Gamma_n$is the set of n-stems in the growing universe'. In a real sense, however, in moving from a poscau process to a covtree process one is losing what one does not actually have. Since, in a walk on labeled poscau, tempting as it is to interpret the node at stage $n$as representing a momentary state of a growing order this is an unphysical picture because the concept of stage $n$has no physical meaning: there is no 'God's eye view' of the universe in asynchronous becoming [17]. Here we see the struggle between locality and global-ness inherent in a gauge theory.

Our aim is to create a covariant framework for two-way in□nite growth and construct the analogue of covtree. The construction of covtree was motivated by the fact that the stem-events exhaust the set of observables in CSG models [10]. Indeed, covtree's algebra of observables is equal to the algebra of stem-events [19]. Therefore, pursuing further the analogy between stems and convex-suborders, in the rest of the paper we introduce and explore a new covariant framework, which we call Z-covtree, whose sample space is Ω^Z and whose set of observables is exactly the set of convex-events. We will see that the structure of Z-covtree is very different from covtree. We will construct Z-covtree via an intermediate construction of a larger tree we call convex-covtree. The next subsection is devoted to defining convex-covtree.

4.1. Defining convex-covtree

Recall that, for any positive integer n, the set of n-orders is called $\Omega(n)$. Let Γ_n denote a subset of $\Omega(n)$. Recall also that an n-convex-suborder means 'a convex-suborder of cardinality n'.

Convex-covtree is a partial order, a directed tree whose nodes at level n are subsets of $\Omega(n)$: a subset $\Gamma_n \subset \Omega(n)$ is a node in convex-covtree if and only if it is the set of n-convex-suborders of some (finite or infinite) order C. In the following, we formalise the definition of convex-covtree.

Definition 4.1. An order C is a certificate of Γ_n if Γ_n is the set of n-convex-suborders of C. A labeled certificate of Γ_n is a representative of a certificate of Γ_n.

A certificate may be finite or infinite, and if it is infinite it may be past-finite, future-finite or two-way infinite. Note that some $\Gamma_n \subset \Omega(n)$ have no certificates at all. If Γ_n has an infinite certificate then it has a finite certificate, but the converse is not true. Examples are shown in figure 7.[18]

18 Note that definition 4.1 of certificate is different to that in [19] where a certificate of Γ_n is an order whose set of n-stems is Γ_n. If C is a certificate of Γ_n by definition 4.1, then C certifies that Γ_n is a node in convex-covtree. If C is a certificate of Γ_n by the definition in [19], then C certifies that Γ_n is a node in covtree. The properties of the certificates depend on which definition of certificate is used, e.g. using the definition in [19] Γ_n has an infinite certificate if and only if it has a finite certificate, while using definition 4.1 the existence of a finite certificate does not imply the existence of an infinite certificate.

Figure 7. Illustration of certificates. C and D are certificates of Γ_3. Γ'_3 has no certificates since any order that contains the three-chain and the three-antichain as three-convex-suborders also contains the ‘L’ order as a three-convex-suborder. E is a certificate of Γ_4. Γ'_4 has no infinite certificates.

Figure 8. Illustration of the operation O^-_c.

We use χ to denote the set of all Γ_n’s, for all n, that have at least one certificate:

$$\chi := \bigcup_{n\in\mathbb{N}} \{\Gamma_n \subseteq \Omega(n) \,|\, \exists \text{ a certificate of } \Gamma_n\}. \quad (22)$$

χ is the ground-set of convex-covtree. To define the partial order on χ, we introduce the map O^-_c:

Definition 4.2. For any n and any set Γ_n of n-orders, the map O^-_c takes Γ_n to the set of $(n-1)$-convex-suborders of elements of Γ_n:

$$O^-_c(\Gamma_n) := \{B \in \Omega(n-1) \,|\, \exists A \in \Gamma_n \text{ s.t. } B \text{ is an } (n-1)\text{-convex-suborder in } A\}. \quad (23)$$

One way to think about the operation O^-_c on Γ_n is to pick an n-order in Γ_n and delete a maximal or minimal element of it to form an $(n-1)$-order. The set $O^-_c(\Gamma_n)$ is the set of all $(n-1)$-orders that can be formed in this way. An illustration is shown in figure 8.

Lemma 4.3. If $\Gamma_n \in \chi$ then $O^-_c(\Gamma_n) \in \chi$.

Proof. There exists a certificate C of Γ_n. Each element of Γ_n is a convex-suborder of C. So each convex-suborder of each element of Γ_n is a convex-suborder of C. An $(n-1)$-order is an $(n-1)$-convex-suborder of C if and only if it is a convex-suborder of some n-convex-suborder of C. Therefore C is a certificate of $O^-_c(\Gamma_n)$. □

Figure 9. The first three levels of convex-covtree.

We can now give the definition of convex-covtree:

Definition 4.4. Convex-covtree is the partial order $(\chi, \prec)$, where $\Gamma_n \in \chi$ is directly above—covers—$O^-_c(\Gamma_n) \in \chi$.

The nodes in the first three levels of convex-covtree are shown in figures 9 and 10. The construction of convex-covtree is closely analogous to the construction of covtree in [19], with the concept of convex-suborder replacing the concept of stem. Indeed, the two resulting structures share some features, including: (1) if C is a certificate of a node Γ_n then C is a certificate of all nodes below Γ_n and (2) every inextendible path has a certificate (as we will prove for convex-covtree in lemma 4.10 and proposition 4.12 below), where the certificate of a path is defined as,

Definition 4.5. An order C is a certificate of a path P if it is a certificate of every node in P.

Properties (1)and(2) allow us to interpret a random walk up convex-covtreeas a covariant process of growth: the growing order is a certi□cate of the path that is taken by the random walk. Each node in the path corresponds to a covariant property of the growing order, i.e. Γ_n is the set of n-convex-suborders of the growing order. At stage n, the walk transitions from the set of (n−1)-convex-subordersof the growing order to the set of n-convex-suborders. At each stage of the random process, more physical information about the growing order is acquired.

4.2. Sample space for convex-covtree

In labeled alternating sequential growth models, there is a 1–1 correspondence between the set of paths on alternating poscau and the set of labeled causets, $\tilde{\Omega}_Z$, and we refer to the latter as the sample space of the process. Events in the event algebra are subsets of this sample space. Covariant events are covariant subsets of this sample space.

The framework of random walks up convex-covtree, is motivated by doing away with mention of labeled causets from the very start. In keeping with this, but keeping to the physical interpretation that the process is producing a growing order, we conceive of the sample space of the process, not as a set of labeled causets, but as a set of orders.

Definition 4.6. The sample space of a random walk on convex-covtree is the set of orders that are certi□cates of inextendible (maximal) paths in convex-covtree.

Figure 10. 22 nodes of convex-covtree and their certi□cates. These are the level 3 nodes

that appear directly above the doublet.

There is no 1–1 correspondence between inextendible paths in convex-covtree and orders: we have already seen this in the example of ATP where almost surely any causal set grown in ATP has the same convex-suborders as any other. So the single path in convex-covtree containing the node at level nthat is the set of all n-orders, for all n, has all the ATP orders as certi□cates. We will come back to this point in the discussion.

Now, we can ask: which orders are in this sample space for walks on convex-covtree? In contrast to all growth models de□ned to date, a walk up convex-covtree can produce □nite orders. This is because convex-covtree contains maximal nodes, so some of its inextendible paths are □nite. A □nite inextendible path has one unique □nite certi□cate, and so if a random walk ends at a maximal element of convex-covtree, then a □nite order is generated and the universe has a beginning and an end. This result and others about maximal nodes and □nite inextendible paths will be proved in the next subsection 4.3. The certi□cates of in□nite paths are necessarily in□nite (since they contain n-convex-suborders for every n>0) and every in□nite order (past-□nite, future-□nite or neither) is a certi□cate of some in□nite path.

In summary, the sample space of a random walk on convex-covtree contains all in□nite orders and many (but not all) □nite orders. It is natural to ask whether there is a way to consistently restrict the sample space to ΩZ, in order that the sample space matches that of the alternating sequential growth models of the previous section. We will show in section 4.4 that this can be done and that in this case the observables are the convex-events. We will also show

that an inconsistency arises ((2) is violated) when restricting the sample space to ΩN, suggesting

that convex-subordersare unsuitable for describing past-□nite growth.

4.3. Finite inextendible paths

Amaximal node is a node that has no descendants—it is maximal in the convex-covtree partial

order. A subset $\Gamma n \subset \Omega(n)$isasingleton if it contains only a single n-order, i.e. $\Gamma n=\{C\}$.Note

that every singleton is a node in convex-covtree since if $\Gamma n=\{C\}$then Cis a certi□cate of Γn.

Lemma 4.7. A maximal node is a singleton $\Gamma n=\{C\}$whose only certi□cate is C.

Proof. Let $\Gamma n=\{C\}$and let its only certi□cate be C. Suppose for contradiction that Γn+1□

Γn. Then there exists some Dwith cardinality >nthat is a certi□cate of Γn+1and hence of Γn.

Contradiction. Therefore Γnis maximal.

Suppose that $\Gamma n=\{C\}$has a certi□cate D□=C.ThenDhas cardinality >nand therefore

$\{D\}$□Γn=⇒Γnis not maximal. Similarly, if Γnis not a singleton then it has a certi□cate D

with cardinality >n=⇒$\{D\}$□Γn.□

The singleton Γ4=(also shown in □gure 7) is an example of a maximal node. To

see that Γ4has no certi□cate of cardinality >4 it is suf□cient to attempt to construct such a

certi□cate by adding a single element to (a representative of) . For example, we can add

the new element to form the □ve-order , but this □ve-order is not a certi□cate of Γ4since it

contains the as a four-convex-suborder. Continuing in this way, we □nd that it is impossible

to form a certi□cate of Γ4by adding an element to . Indeed, is the unique certi□cate of

Γ4.

The existence of maximal nodes implies the existence of □nite inextendible paths. We can

characterise □nite inextendible paths as follows:

Proposition 4.8. An inextendible path Pis □nite if and only if it contains a singleton $\{Cn\}$,

where Cnis not the n-chain or the n-antichain.

To prove proposition 4.8 we will need:

Lemma 4.9. Let Cnbe an n-order that is not the n-chain or the n-antichain. Then every

certi□cate of $\{Cn\}$has cardinality less than n2.

Proof. For any (□nite or in□nite) order C,letw(C)andh(C) denote the width and height

of C, respectively. Note that |C|□h(C)w(C). Additionally, if Cis a certi□cate of $\{Cn\}$then

w(C)=w(Cn)<n. We will show that if Cis a certi□cate of $\{Cn\}$then h(C)□nand the result

follows.

Let Cbe an order with h(C)>nand suppose for contradiction that Cis certi□cate of $\{Cn\}$.

Let D be a chain of length $n+1$ in C and let H be the convex hull of D. Then $|H| = n+k$ for some $k>0$. Note that H is an interval by construction, i.e. it has a single maximal element and a single minimal element. We will now show by induction that H is a chain and therefore C is not a certi□cate of $\{C_n\}$.

One way to obtain C_n from H is to remove the minimal element of H to form the order H_{-1}, then remove a minimal element of H_{-1} to form H_{-2} and so on until $H_{-k}=C_n$. Since H has a unique maximal element, $H_{-k}=C_n$ has a unique maximal element.

Another way to obtain C_n from H is to remove the maximal element of H to form the order H^{-1}, then remove a minimal element of H^{-1} to form H^{-1}

-1, then remove a minimal element of

H^{-1}

-1 to form H^{-1}

-2 and continue to remove minimal elements until H^{-1}

$-k+1=C_n$. The top level

of H^{-1}

$-k+1=C_n$ is level $h(C)-1$ of H, and since C_n has a unique maximal element we learn that H has only one element at level $h(C)-1$.

Suppose H has only one element at each of the levels $h(C), h(C)-1, ..., h(C)-r+1$ for some $r<h(C)$. Then H^{-r}

$-k+r=C_n$ is constructed by removing the top r levels of H and there-

fore the top level of H^{-r}

$-k+r=C_n$ is level $h(C)-r$ of H. Since H^{-r}

$-k+r=C_n$ has a unique max-

imal element we learn that H has only one element at level $h(C)-r$. Therefore, by induction H has a single element at each level, i.e. H is a chain. □

Proof to Proposition 4.8. Let $\{C_n\} \in P$ and suppose for contradiction that P is infinite. Then for any $N > n^2$ there exists a node $\Gamma_N \in P$. Let C denote a certificate of Γ_N and note that $|C| \geq N > n^2$. Since $\Gamma_N \subseteq \{C_n\}$, C is a certificate of $\{C_n\}$. Contradiction. That the converse is true follows from the fact that every maximal node is a singleton (lemma 4.7). □

We can also identify the certificates of the finite inextendible paths:

Lemma 4.10. If $P = \Gamma_1 \prec \Gamma_2 \prec .. \prec \Gamma_k$ is a finite inextendible path then $C_k \in \Gamma_k$ is the unique certificate of P.

Proof. Clearly, C_k is a certificate of P and there are no other certificates of P with cardinality $\leq k$. Suppose C_l is a certificate of P with cardinality $l > k$. Then $\{C_l\} \subseteq \Gamma_k$. Contradiction. □

A corollary is that the corresponding sample space contains spacetimes of finite volume, namely the certificates of the finite inextendible paths. An n-order C_n is an element of the sample space if there is no order $D \neq C_n$ whose only n-convex-suborder is C_n. For example, the sample space contains the four-order , but it does not contain the 'L' order, , since

.

Lemma 4.11. The sample space contains countably many finite orders.

Proof. Let $Q(n)$ denote the number of singletons $\{C_n\}$ at level n in convex-covtree, where C_n is not the n-chain or the n-antichain. Each of these $Q(n)$ nodes is in at least one finite path and no two are in the same path. Therefore there are at least $\lim_{n\to\infty} Q(n)$ finite inextendible paths. □

It may seem that the sample space is entropically dominated by the infinite orders, as there are uncountably many of these and only countably many finite orders. But if one assigns transition probabilities uniformly such that the probabilities to transition from a given node of convex-covtree to any of its children are equal, then the event that spacetime has finite cardinality happens with probability $> \frac{1}{2^2}$ (since this is the probability of reaching a singleton that does not contain a chain or an antichain by level 3). By proposition 4.8 the models which almost surely produce infinite universes are exactly those that satisfy $P(\Gamma) = 0$ whenever Γ is a singleton node that does not contain a chain or an antichain.[19]

4.4. Infinite paths and Z-covtree

We now prove that:

Proposition 4.12. Every infinite path in convex-covtree has a certificate.

Together, lemma 4.10 and proposition 4.12 enable us to interpret a walk on convex-covtree as a process in which an order grows—they guarantee that each realisation of the walk will

19 For any $n>1$, if Γ_n is a singleton that contains a chain then it is contained in a unique inextendible path,

. Similarly, if Γ_n is a singleton that contains an antichain then it is contained in a unique inextendible path, .

produce some order. A path has more than one certificate if its certificates are convex-rogues and, in this case, which certificate is the growing order is up for interpretation (e.g. we can consider all certificates of a given path to be physically equivalent).

To prove proposition 4.12, we adapt the algorithm from [19] that generates a certificate for any infinite path P. We will need the concept of 'minimal certificates':

Definition 4.13. Given some Γ_n, we order its finite certificates by inclusion. Let C_1 and C_2 be two finite certificates of Γ_n. We say C_1 □ C_2 if and only if C_1 is a convex-suborder in C_2. A minimal certificate of Γ_n is minimal in this order.

We will also need the following lemma:

Lemma 4.14. Let $P=\Gamma_1 \prec \Gamma_2 \prec \Gamma_3 \prec \ldots$ be an infinite path in convex-covtree. Then for any $\Gamma_n \in P$ there exists some $\Gamma_m \in P$ that contains a certificate of Γ_n.

Proof. First, note that it follows from the definition of convex-covtree that if Γ_n is a singleton and Γ_m □ Γ_n then any $C \in \Gamma_m$ is a certificate of Γ_n. If Γ_n is not a singleton, then every minimal certificate C of Γ_n satisfies $n<|C|\leq N$ where $N:=n|\Gamma_n|$. Consider $\Gamma_N\in P$ and let D be a finite certificate of Γ_N. Since a certificate of a node is a certificate of all the nodes below it, D is a certificate of Γ_n. Now, at least one minimal certificate of Γ_n occurs as a convex-suborder in D. Choose one, call it C, let $m:=|C|$ and consider $\Gamma_m\in P$. Γ_m is the set of all convex-suborders of cardinality m in D and so C is an element of Γ_m. □

Proof of Proposition 4.12. Given an infinite path $P=\Gamma_1\prec\Gamma_2\prec\ldots$, the following inductive algorithm generates an infinite nested sequence of causal sets, $\tilde{C}_{m_1}\subset\tilde{C}_{m_2}\subset\tilde{C}_{m_3}\subset\ldots$:

Step 1:

(1.0) Pick some natural number $m_0>0$ and consider $\Gamma_{m_0}\in P$.

(1.1) By lemma 4.14, there exists some $\Gamma_{m_1}\in P$ that contains some certificate C_{m_1} of Γ_{m_0}. Pick a representative $\tilde{C}_{m_1}$ of C_{m_1}.

(1.2) Go to step 2.

Step $k>1$:

(k.1) By lemma 4.14, there exists some $\Gamma_{m_k}\in P$ that contains some certificate C_{m_k} of $\Gamma_{m_{k-1}}$. Pick a representative $\tilde{C}_{m_k}$ of C_{m_k} such that $\tilde{C}_{m_{k-1}}$ from the previous step is a sub-causet of $\tilde{C}_{m_k}$.

(k.2) Go to step $k+1$.

By construction, the union $\tilde{C}:=\bigcup_{i=1}^{\infty}\tilde{C}_{m_i}$ is order-isomorphic to a labeled certificate of P. If the ground-set of $\tilde{C}$ is $\mathbb{Z}$, $\mathbb{N}$ or $\mathbb{Z}^-$ then $\tilde{C}$ is a labeled certificate of the path P. If $\tilde{C}$ has ground-set $[p,\infty)$ for some $p \in \mathbb{Z}$ then $\tilde{C}$ is order-isomorphic to some causet $\tilde{D}$ with ground-set $\mathbb{N}$. In this case $\tilde{D}$ is a labeled certificate of P. If $\tilde{C}$ has ground-set $(-\infty,p]$ for some $p \in \mathbb{Z}$ then $\tilde{C}$ is order-isomorphic to some causet $\tilde{E}$ with ground-set $\mathbb{Z}^-$. In this case $\tilde{E}$ is a labeled certificate of P. Since in each case P has a labeled certificate, every infinite path has a certificate. □

As previously stated, the sample space of convex-covtree contains all infinite orders and countably many (but not all) finite ones. We set out to find a covariant counterpart to alternating sequential growth, and now we see that convex-covtree is not that framework. We now ask: can convex-covtree can be truncated into a tree whose sample space equals $\Omega_\mathbb{Z}$?

For a start, we can consider the subtree of convex-covtree that contains only the nodes that have infinite certificates or equivalently the subtree of convex-covtree that is the union of all infinite paths. By truncating the finite inextendible paths we remove the finite orders from the sample space and proposition 4.12 guarantees that each inextendible path in this truncated covtree has a certificate in Ω.

However, there is no guarantee that every path has a certi□cate in $\Omega_{\mathbb{Z}}$. Indeed, there exist in□nite paths that only have certi□cates in $\Omega_{\mathbb{N}}$and others that only have certi□cates in $\Omega_{\mathbb{Z}^-}$. Recall that a certi□cate of a path is a certi□cate of all its nodes. Therefore if there exists some $\Gamma_n\in P$whose in□nite certi□cates are only in $\Omega_{\mathbb{N}}$then Ponly has certi□cates in $\Omega_{\mathbb{N}}$.

For example, consider the node whose unique minimal certi□cate is .We can construct any certi□cate of $\Gamma_3$by starting with its minimal certi□cate and then adding elements to it. In particular, if $\Gamma_3$has a certi□cate in $\Omega_{\mathbb{Z}}$or $\Omega_{\mathbb{Z}^-}$then we should be able to grow a certi□cate of $\Gamma_3$by adding an element that is spacelike or to the past of every element in . There are 5 ways to add such an element, but none produces a certi□cate of Γ_3(e.g. contains the three-antichain as a convex-suborder). Therefore, $\Gamma_3$has no certi□cates in $\Omega_{\mathbb{Z}}$or in $\Omega_{\mathbb{Z}^-}$. Finally, note that $\Gamma_3$does have a certi□cate in $\Omega_{\mathbb{N}}$, namely the order that contains the topped with an in□nite chain. Therefore the in□nite path containing $\Gamma_3$only has certi□cates in $\Omega_{\mathbb{N}}$. Similarly, if there exists some $\Gamma_n\in P$ all of whose in□nite certi□cates are in $\Omega_{\mathbb{Z}^-}$then Ponly has certi□cates in $\Omega_{\mathbb{Z}^-}$(see for example the node and the in□nite path that contains it).

The following proposition identi□es the paths that have certi□cates in $\Omega_{\mathbb{Z}}$and which are therefore of interest to us,

Proposition 4.15. An in□nite path Phas a certi□cate in $\Omega_{\mathbb{Z}}$if and only if every node in P has a certi□cate in $\Omega_{\mathbb{Z}}$.

Proof. Given an in□nite path $P=\Gamma_1\prec\Gamma_2\prec\ldots$ each of whose nodes has a certi□cate in $\Omega_{\mathbb{Z}}$, the following inductive algorithm generates an in□nite nested sequence of causal sets, $\tilde{C}_{t_1}\subset\tilde{C}_{t_2}\subset\ldots$, whose ground-sets $[r_1,s_1]$, $[r_2,s_2]$, ...respectively, satisfy $r_1>r_2>\ldots$ and $s_1<s_2<\ldots$:

Step 1:

(1.0) Pick some natural number $m_0>0$ and consider $\Gamma_{m_0}\in P$.

(1.1) By lemma 4.14, there exists some $\Gamma_{m_1}\in P$ that contains some certificate C_{m_1} of Γ_{m_0}. Pick a representative $\tilde{C}_{m_1}$ of C_{m_1} and set $\tilde{C}_{t_1} := \tilde{C}_{m_1}$.

(1.2) Go to step 2.

Step $k>1$:

(k.1) By lemma 4.14, there exists some $\Gamma_{m_k}\in P$ that contains some certificate C_{m_k} of $\Gamma_{t_{k-1}}\in P$. Additionally, there exists a representative $\tilde{C}_{m_k}$ of C_{m_k} with ground-set $[p_k,q_k]$ that contains $\tilde{C}_{t_{k-1}}$ as a sub-causet and satisfies at least one of (a) $p_k<r_{k-1}$ or (b) $q_k>s_{k-1}$. If there exists some $\tilde{C}_{m_k}$ that satisfies both (a) and (b), set $\tilde{C}_{t_k}:=\tilde{C}_{m_k}$. Otherwise, pick a representative $\tilde{C}_{m_k}$ that satisfies (a) or (b). Go up one node along the path to $\Gamma_{1+m_k}\in P$. Let $\tilde{C}\in\tilde{\Omega}_{\mathbb{Z}}$ be an infinite certificate of Γ_{1+m_k} that contains $\tilde{C}_{m_k}$ as a subcauset. Set $\tilde{C}_{t_k}:=\tilde{C}|[p_k,q_k+1]$ if $\tilde{C}_{m_k}$ satisfies (a) or $\tilde{C}_{t_k}:=\tilde{C}|[p_k-1,q_k]$ if $\tilde{}$

Cmksatis□es (b).

(k.2) Go to step k+1.

By construction, the union ˜

C:=□∞

i=1˜

Cti∈˜

ΩZis a labeled certi□cate of P. Therefore, if

every node in Phas a certi□cate in ΩZthen Phas a certi□cate in ΩZ. That the converse is true follows from de□nition 4.5.□

Finally, we can de□ne:

Definition 4.16. Z-covtree is the subtree of convex-covtree that contains exactly all nodes that have a certi□cate in ΩZ.

Z-covtree is the two-way in□nite analogue of covtree that we have set out to build. Proposition 4.15 guarantees that every inextendible path in Z-covtree has at least one certi□cate in ΩZand thus allows for every random walk on Z-covtree to be interpreted as a dynamics with sample space ΩZ. To see the relationship between a walk on Z-covtree and the corresponding dynamics, for each Γnin Z-covtree let certZ(Γn)⊂˜

ΩZdenote the set of

labeled certi□cates of Γnwhose ground set is Z.LetΣbe the σ-algebra generated by all the certZ(Γn)'s. A dynamics is then the probability measure space (ΩZ,Σ,P) where the measure Pis given by P(certZ(Γn)) =P(Γn). We will now show that the observables of these dynamics (i.e. the elements of Σ) are the convex-events.

Recall that, for each □nite order Cn, convex(Cn)⊂˜

ΩZis the collection of causets that contain C_n as a convex-suborder. Let $R(C)$ denote the σ-algebra generated by the convex(C_n)'s. A convex-event is an element of $R(C)$.[20]

Lemma 4.17. $\Sigma = R(C)$.

Proof. We will show that any convex(C_n) can be constructed by finite set operations on the certZ(Γ_m)'s and vice versa, and the result follows.

Consider an n-order B_n. Let Γ^i_n be the nodes in convex-covtree that contain B_n, where i labels the individual nodes. Suppose $E \in$ certZ(Γ^i_n) for some i. Then B_n is an n-convex-suborder in E and hence $E \in$ convex(B_n). Suppose $E \notin$ certZ(Γ^i_n) for all i. Then B_n is not an n-convex-suborder in E and hence $E \notin$ convex(B_n). It follows that convex(B_n) $= \bigcup_i$ certZ(Γ^i_n).

Consider some node $\Gamma_n = \{A^1_n, ..., A^k_n\}$ in convex-covtree. Let $\Omega(n) \setminus \Gamma_n = \{B^1_n, ..., B^l_n\}$.

Suppose $E \in$ certZ(Γ_n). Then $A^1_n, ..., A^k_n$ are n-convex-suborders in E, and $B^1_n, ..., B^l_n$ are not n-convex-suborders in E. Hence $E \in \bigcap_{i=1}^k \text{convex}(A^i_n) \setminus \bigcup_{j=1}^l \text{convex}(B^j_n)$. Suppose $E \notin$ certZ(Γ_n). Then either (i) there exists some $A^i_n \in \Gamma_n$ that is not an n-convex-suborder in E=⇒E/∈□k

i=1convex(Ai

n), or (ii) there exists some Bj

n∈Ω(n)\Γnthat

is an n-convex-suborder in E=⇒E∈□l

j=1convex(Bj

n). It follows that, certZ(Γn)=

□k

i=1convex(Ai

n)\□l

j=1convex(Bj

n). □

Lemma 4.17 strengthens the analogy between covtree and Z-covtree—the observables of

covtree are the stem-events while the observables of Z-covtree are the convex-events.Z-covtree

is to alternating poscau what covtree is to labeled poscau. Convex-suborders are to two-way

in□nite dynamics what stems are to past-□nite dynamics.

5. Discussion

In this work, we set out to build frameworks for growth dynamics for two-way in□nite causal

sets. We began by adapting the sequential growth paradigm to create alternating growth mod-

els. We discussed the dif□culties in attributing any physical signi□cance to the process of

alternating growth and dif□culties in formulating and interpreting a ‘causality’ condition in

this framework. We showed that the only alternating CSG model that satis□es DGC is ATP.

These may be considered as evidence against the existence of physically meaningful dynamical

growth models for two-way in□nite causal sets.

20 It may seem that labeled causets have snuck back into the story. However, though in section 3we formally de□ned a

convex-event to be a set of labeled causets, because the de□nition of convex(Cn) is label independent, the convex(Cn)’s

and the convex-events generated by them are covariant and can be thought of —in the obvious way— as subsets of

ΩZ—i.e. sets of orders.

Figure 11. The two-way in□nite comb (left) and the future in□nite comb above an in□nite chain (right) have the same n-convex-suborders for every n>0, therefore every convex-event contains either both or neither. The order on the right contains posts while the order on the left does not. Therefore, the event that the completed order contains a post is not a convex-event.

On the positive side, we identi□ed a set of covariant observables that possess a clear physical interpretation, namely the convex-events. However we also showed that that ATP is deterministic with respect to the convex-events: the probability of any convex-event in ATP is 0 or 1 and in particular the probability of any □nite order being a convex-suborder of the growing causet is 1. There do exist alternating CSG models for which this is not the case, suggesting that there may be models in which the convex-events may yet form a rich and interesting class of observables. This depends on future developments and whether some physically motivated and interesting alternating sequential growth models can be found.

We then used the notion of convex-suborders and convex-events to adapt the covariant growth framework of [19] to two-way in□nite growth. We encountered additional complications that are not present in the original construction, namely that the existence of a □nite certi□cate does not guarantee the existence of an in□nite certi□cate and that the existence of an in□nite certi□cate does not guarantee the existence of a certi□cate in ΩZ. Nevertheless, we were able to de□ne a consistent covariant framework for two-way growth, Z-covtree, whose observables are the convex-events.

Throughout, we were led to considering convex-suborders as the basic physical properties for two-way in□nite growth by pursuing an analogy with stems and the role that they play in past-finite growth.In fact, convex-suborders are a generalisationof stems—a stem is a convex-suborder that contains its own past.21 Nevertheless, there may be other entities that could be considered as physical properties for two-way infinite dynamics, for example, downsets (subcausets that contain their own past—a generalisation of stem in which the condition of finite cardinality is relaxed), moment of time surfaces (thickened antichains [34]), or intervals (special cases of convex-suborders). While these alternatives may prove fruitful in the future, we can identify a property unique to convex-suborders that is essential for our constructions: every infinite order contains at least one n-convex-suborder for every $n>0$.

A significant downside of our new covariant framework is that the event that the completed order contains a post is not measurable since it is not a convex-event(figure 11). Moreover, the cosmic renormalisation transformation associated with posts relies crucially on the cardinality of the past of the post, while a post in a two-way infinite order will necessarily have an infinite

21 When considering both the convex(C_n)'s and the stem(C_n)'s as subsets of $\tilde{\Omega}_N$, a convex-event is a special case of a stem-event.

past. Both posts and cosmic renormalisation play a pivotal role in the conception of causal set cosmology [23,27,33] and so the two-way infinite growth models for causal set cosmology will require a new way of thinking about this cosmological paradigm.

Another challenge is to identify alternating CSG dynamics in which there is a large and rich enough class of convex-eventsthat serve usefully to discriminate between different realisations of the process, including with measures that lie strictly between 0 and 1. To this end we may need to consider the sequence (p_n), a representation of the CSG models that is related to the

t_k's by equation (19). When (p_n) is a constant sequence, the dynamics is TP and the measure of every convex-event is equal to 1. What behaviour does the sequence (p_n) need to display in order for a dynamics to be probabilistic with respect to convex-events? How quickly must the sequence (p_n) increase or decrease to give sufficiently different behaviour from the constant sequence of TP?

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Data availability statement

No new data were created or analysed in this study.

Appendix A. Table of symbols defined in the text

See table 1.

Appendix B. On infinite certificates of nodes and paths in convex-covtree

By proposition 4.12, every infinite path in convex-covtree has at least one certificate in Ω. By proposition 4.15, an infinite path in convex-covtree has a certificate in Ω_Z if and only if each of its nodes has a certificate in Ω_Z. There exist nodes whose infinite certificates are only contained in Ω_N or only in Ω_{Z^-} (see section 4.4 for examples), and therefore the infinite paths containing these nodes only have certificates in Ω_N or in Ω_{Z^-}, respectively.

There exists no node in convex-covtree whose infinite certificates are only contained in Ω_Z, since if a node has a certificate in Ω_Z then it has a certificate in Ω_N and in Ω_{Z^-}. To see this, let ˜

C∈˜

ΩZbe a labeled certi□cate of some Γnand let ˜

C|[k,l]be a □nite certi□cate of Γn.Then

˜

C|[k,∞)is order-isomorphic to some ˜

D∈˜

ΩNand ˜

Dis a certi□cate of Γn. Similarly, ˜

C|(∞,l]is

order-isomorphic to some ˜

E∈˜

ΩZ−and ˜

Eis a certi□cate of Γn.

Tab le 1. Table of symbols de□ned in the text.

˜

C,˜

D,... Labeled causets

C,D,... Orders

∼

=˜

C∼

=˜

Dif ˜

Cand ˜

Dareequaluptoanorder-isomorphism

˜

ΩNThe set of labeled causets with ground-set N

˜

ΩZThe set of labeled causets with ground-set Z

˜

ΩZ−The set of labeled causets with ground-set Z−

˜

ΩThe set of in□nite labeled causets, ˜

Ω≡˜

ΩN□˜

ΩZ□˜

ΩZ−

ΩThe set of in□nite orders, Ω:=˜

Ω/∼

=

ΩNThe set of orders that have a representative in ˜

ΩN

ΩZThe set of orders that have a representative in ˜

ΩZ

ΩZ−The set of orders that have a representative in ˜

ΩZ−

Ω(n)Thesetofn-orders for some n∈N+

ΓnA subset of Ω(n)

Figure 12. The order $D \in \Omega_{\mathbb{Z}}$ shown on the right is a certificate of the path P. Every node in P has a certificate in $\Omega_{\mathbb{N}}$: D_3 is a certificate of $\Gamma_n \in P$ only for $n \geq 3$, D_4 is a certificate of $\Gamma_n \in P$ only for $n \geq 4$, D_5 is a certificate of $\Gamma_n \in P$ only for $n \geq 5$, etc. There is no order in $\Omega_{\mathbb{N}}$ that is a certificate of every node in P.

There exist infinite paths in convex-covtree whose infinite certificates are only contained in $\Omega_{\mathbb{Z}}$. An infinite path only has certificates in $\Omega_{\mathbb{Z}}$ if and only if there is no one order in $\Omega_{\mathbb{N}} \cup \Omega_{\mathbb{Z}^-}$ that is a certificate of every node in the path. For example, consider the path whose certificate is the order D shown on the right of figure 12. Each node in P has a certificate in $\Omega_{\mathbb{N}}$, as illustrated in figure 12, but there is no order in $\Omega_{\mathbb{N}}$ that is a certificate of every node in P. One way to see this is to notice that for every $n>3$, $\Gamma_n \in P$ has a unique minimal certificate, namely the diamond sandwiched between two $(n-3)$-chains. Now, pick some $n>3$ and w.l.g. pick a representative of its minimal certificate, $\tilde{C}_{2n-6}$, with ground-set $[0, 2n-6]$. We seek a labeled minimal certificate $\tilde{C}_{2n-4}$ of Γ_{n+1} that contains $\tilde{C}_{2n-6}$ as a subcauset, and find that $\tilde{C}_{2n-4}$ must have ground-set $[-1, 2n-5]$. Next we seek a labeled minimal certificate $\tilde{C}_{2n-2}$ of Γ_{n+2} that contains $\tilde{C}_{2n-4}$ as a subcauset, and find that $\tilde{C}_{2n-2}$ must have ground-set $[-2, 2n-4]$ etc. Since at each stage we add a positive and a negative integer to the ground-set, in the infinite limit the labeled certificate must have ground-set $\mathbb{Z}$.

Since the existence of a certificate in $\Omega_{\mathbb{N}}$ for each $\Gamma_n \in P$ does not guarantee that P has a certificate in $\Omega_{\mathbb{N}}$ (i.e. there is no analogue of proposition 4.15 for $\Omega_{\mathbb{N}}$) there is no subtree of convex-covtree that contains exactly all infinite paths that have certificates in $\Omega_{\mathbb{N}}$, i.e. there is no $\mathbb{N}$ analogue of Z-covtree. Thus, convex-covtree cannot be truncated into a growth framework whose sample space is $\Omega_{\mathbb{N}}$, suggesting that convex-events (now treated as subsets of $\Omega_{\mathbb{N}}$) are not rich enough to exhaust the set of observables in past-finite dynamics.

One can understand this difference between $\Omega_{\mathbb{N}}$ and $\Omega_{\mathbb{Z}}$ using metric space techniques. For any two orders C and D, let $C \sim D$ if and only if C and D are a convex-rogue pair, i.e. if they share the same n-convex-suborders for all n. Let $\Omega_{\mathbb{N}}/\sim$ and $\Omega_{\mathbb{Z}}/\sim$ be quotient spaces under the convex-rogue equivalence relation, so that their elements are equivalence classes of orders denoted by $[C]$ etc. We can consider these quotient spaces as metric spaces with metric $d([C], [D]) = \frac{1}{2^n}$, where n is the largest integer for which representatives of $[C]$ and $[D]$ have the same sets of n-convex-suborders. Given a node Γ_n in convex-covtree we can associate with it a subset $[\mathrm{cert}_{\mathbb{N}}(\Gamma_n)] \subseteq \Omega_{\mathbb{N}}/\sim$, namely the set of elements of $\Omega_{\mathbb{N}}/\sim$ whose representatives are certificates of Γ_n, and similarly $[\mathrm{cert}_{\mathbb{Z}}(\Gamma_n)] \subseteq \Omega_{\mathbb{Z}}/\sim$. Given a path $P = \Gamma_1 \prec \Gamma_2 \prec \ldots$, we can associate with it the sets $[\mathrm{cert}_{\mathbb{N}}(P)] = \bigcap_{\Gamma_n \in P} [\mathrm{cert}_{\mathbb{N}}(\Gamma_n)]$ and $[\mathrm{cert}_{\mathbb{Z}}(P)] = \bigcap_{\Gamma_n \in P} [\mathrm{cert}_{\mathbb{Z}}(\Gamma_n)]$. Since the metric space $(\Omega_{\mathbb{Z}}/\sim, d)$ is complete, by Cantor's lemma $[\mathrm{cert}_{\mathbb{Z}}(P)]$ is non-empty whenever all the $[\mathrm{cert}_{\mathbb{Z}}(\Gamma_n)]$'s are non-empty (cf proposition 4.15). On the other hand, the metric space $(\Omega_{\mathbb{N}}/\sim, d)$ is not complete and therefore $[\mathrm{cert}_{\mathbb{N}}(P)]$ can be empty when all the $[\mathrm{cert}_{\mathbb{N}}(\Gamma_n)]$'s are non-empty.